JIM BRYAN

QUANTUM INVARIANTS OF CALABI-YAU THREE-FOLDS

LECTURE NOTES BY POOYA RONAGH

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The following are lecture notes from a course offered at the University of British Columbia (UBC), in spring 2014 by Prof. J. Bryan. The script at hand is compiled from lecture notes I took in the class, and are typeset in Tufte book style, TUFTE-LATEX.GOOGLECODE.COM.

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Introduction

This course is about the quantum invariants themselves rather than the geometry of the underlying spaces. In other words, we are not necessarily going to learn more about the Calabi-Yau threefolds but the goal is really focusing on the very invariants defined for them. We want to:

1. Define these invariants;

2. Study their structure (it turns out that these invariants have very interesting internal structures);

3. Study the relationship between different kinds of invariants; and,

4. Study their connections with other branches of mathematics (one of the beauties of these invariants is that they turn out to be tied to some surprising other topics such as modular forms and representations of algebraic groups).

That said, the first step is to define the Calabi-Yau manifolds:

Definition 0.1. A Calabi-Yau manifold of dimension n is a complex projective manifold[1] $X \subseteq \mathbb{CP}^N$ (a nonsingular projective variety) with $\dim_{\mathbb{C}} X = n$ satisfying any of the following equivalent conditions:

1. X admits a Kähler metric whose Ricci curvature is zero (this is called a Ricci-flat or Calabi-Yau metric)[2]
2. X has a non-vanishing holomorphic n-form;
3. The canonical line bundle of X, $K_X = \bigwedge^n T_X^*$ is trivial: $K_X \cong \mathcal{O}_X$.

Remark 0.2. (1) implying (2) and equivalence of (2) and (3) is easy. (3) implying (1) is Yau's field medal winner theorem.

Remark 0.3. As algebraic geometers we often the (3) as the definition of a Calabi-Yau, even in the case where X is not compact (so when X is quasi-projective). In this non-compact case (3) does not imply (1) but we may still consider X to be Calabi-Yau whenever $K_X \cong \mathcal{O}_X$.

[1] We are always working over the field of complex numbers in this course.

[2] This is a purely geometric condition.

Remark 0.4. Some people might require that $H^1(X, \mathcal{O}_X) = 0$ maybe excluding the case of $\dim X = 1$. (1) does imply $H^1(X, \mathcal{O}_X) = 0$.

Example 0.5. In $\dim_X = 1$ then Calabi-Yau manifolds are elliptic curves, i.e. Riemann surfaces of genus 1 which all turn out to be $\mathbb{C}/\mathbb{Z}^2$ quotients, and i.e. cubic plane curves $X_{(3)} \subseteq \mathbb{CP}^2$ and $H^1(X, \mathcal{O}_X) \cong \mathbb{C}$. There is only one topological type for these (that of a topological torus) but they have a moduli as a complex manifolds.

Example 0.6. In $\dim_X = 2$, Calabi-Yau manifolds are either Abelian surfaces (described as $\mathbb{C}^2/\mathbb{Z}^4$, topologically a $T^{(4)} = (S^1)^4$) or a K3-surface (e.g. a hyper surface of degree 4 in 3-projective plane: $X_{(4)} \subseteq \mathbb{CP}^3$. Abelian surfaces have $H^1(X, \mathcal{O}_X) \neq 0$ but K3 surfaces have $H^1(X, \mathcal{O}_X) = 0$. There are two topological types, each type has a moduli.

Example 0.7. There is a vast number of distinct topological types of CY3s (possibly infinite, conjectured to be finite). About 5 hundred thousand already known!! A quintic threefold $X_{(5)} \subseteq \mathbb{CP}^4$ in an example of a Calabi-Yau threefold.

Example 0.8. More generally, any hypersurface, $X_{n+1} \subseteq \mathbb{CP}^n$, of degree $n+1$ is a Calabi-Yau of dimension $n-1$.

$$X_{(n+1)} = \{(x_0, \cdots, x_n) \in \mathbb{P}^n : F(x_0, \cdots, x_n) = 0\}$$

where F is a homogeneous polynomial of degree $n+1$.

Example 0.9. Generalizing further, if s is a transverse section of the dual canonical bundle $K_M^* = \bigwedge^{d-M} T_M$ of a smooth projective manifold M then $s^{-1}(0)$ is a Calabi-Yau. (This generalizes last example if $M = \mathbb{CP}^n$ and consequently $K_M^* \cong \mathcal{O}(n+1)$, hence sections being given by polynomials of degree $n+1$.)

Example 0.10. For a construction of non-compact examples, let M be a projective manifold of dimension d and $E \to M$ be a holomorphic vector bundle of rank r with $\wedge^n E \cong K_M$, then let $X = \mathrm{tot}(E)$ total space of the vector bundle is a Calabi-Yau of dimension $r+d$.

0.1 *Quantum invariants of Calabi-Yau threefolds*

The quantum invariants of Calabi-Yau threefolds are deformation invariants which have analogous quantities in string theory/QFT.

A deformation invariant is a quantity (usually a number) associated to X which is invariant under deformations of the complex structure of X.

Specific of Calabi-Yau threefolds is that the expected dimension of the space $\mathcal{M}(X, \beta)$ of curves in class β is zero.

The invariants we study arise form counting (in the ideal setting) holomorphic curves $C \subset X$. A curve $C \subset X$ gives rise to a homology class $[C] \in H_2(X; \mathbb{Z})$ and in particular we are interested in counting curves $C \subset X$ in a fixed homology class $\beta \in H_2(X; \mathbb{Z})$. The key point here is that

> The expected dimension of the space $\mathcal{M}(X, \beta)$ of curves $C \subset X$ with $[C] = \beta$ is zero. This is unique to Calabi-Yau threefolds.

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Last time we defined Calabi-Yau manifolds. We will define quantum invariants of Calabi-Yau threefolds now. These will be deformation invariants related to quantities in string theory. Our invariants will arise from counting homolorphic curves $C \subseteq X$. Typically we fix $\beta \in H_2(X; \mathbb{Z})$ and count curves $C \subseteq X$ such that $[C] = \beta$.

To perform this count we will want to construct a moduli space $\mathcal{M}(X, \beta)$ parametrizing curves in the class β. In other words each point in $\mathcal{M}(X, \beta)$ corresponds to a curves and in ideal situation there is a finite number of curves and hence $\mathcal{M}(X, \beta)$ is a zero dimensional projective manifold.

As we will see this ideal situation almost never holds. But we will see how we can get close to having this situation being the case. If X is a Calabi-Yau threefold, then the *expected dimension* of $\mathcal{M}(X, \beta)$ is always zero (regardless of X and β). This is one of the reasons why Calabi-Yau threefolds are special.

0.2 *Expected dimension*

If a variety M is given by the zero locus of a set of equations on some ambient smooth space V, then the expected dimension in this situation is $\dim V - \#\{\text{ equation }\}$. If the solutions to the equations intersections transversally then M is a smooth manifold of dimension equal to the above number. But otherwise (i.e. when M has excess intersection), M may be singular, and it may be larger than expected dimension. So vaguely speaking, the main issues to overcome are:

1. Deal with the excess dimension of moduli space.
2. We will need compactness or projectivity (i.e. a *compactification*).

Different invariants arise from different ways of dealing with these issues.

Blue print of a curve-counting theory

$$X \xrightarrow{\text{handing issue (1)}} \overline{\mathcal{M}}(X, \beta) \xrightarrow{\text{handling issue (2)}} \text{invariants}$$

Calabi-Yau threefold	Some compact moduli space of curves $[C] \subseteq X$ with $[C] = \beta$	Usually the degree of a virtual class, and sometimes dimension of a cohomology group of this space

There are two basic strategies:

1. View $C \subset X$ as being parametrized $f : C \to X$ and keep C as non-singular as possible, but let the map no longer be an embedding.
   This leads to the moduli space of stable maps in Gromov-Witten theory. In physics, these correspond to world sheets of strings (where a one-dimension string travels in time to sweep out a two-dimension Riemannian surface called a world sheet).
2. View $C \subseteq X$ as being cut out be equations. Hence the curves are describe in therm of sheaves, giving rise to various moduli spaces of sheave.
   (a) Moduli space of ideal sheaves: these lead to Donaldson-Thomas invariants.
   (b) Moduli space of stable pairs: these lead to Pandharipande-Thomas theory.
   (c) Moduli space of torsion sheaves: these lead to Gupakumar-Vafa invariants.

   Sheaves (pairs, etc.) in physic are configurations of D-branes (These are boundary conditions in string theory).

0.3 *Syllabus*

The invariants Gromov-Witten (GW), Donaldson-Thomas (DT), Pandharipande-Thomas (PT), and Gupakumar-Vafa (GV) are the main topics of the course. We want to talk about

1. Their definitions;
2. Their structures;
3. And the relationship between them.

The outline of the course is hence as follows:

1. Invariants from curve counting:
   (a) GW invariants (2 weeks)
   (b) DT invariants (2 weeks)
   (c) GW/DT correspondence (2 weeks)
   (d) PT theory and PT/DT correspondence (2 weeks)
   (e) GV theory and GV conjecture (1 week)
2. Computational methods of the invariants
   (a) Localization and topological vertex
   (b) Degeneration and relative invariants (?)
   (c) Motivic methods
3. Local geometries
   (a) Local curves (this leads to the interesting TQFT)
   (b) Local surfaces and Göttsche conjectures
4. Orbifold Calabi-Yau threefolds and their resolutions
5. Refined invariants and wall-crossing techniques.

0.4 Course references

A nice overview on curve-counting invariants is written by Pandharipande and Thomas [3]. For Gromov-Witten invariants many sources such as the relevant chapters of Mirror Symmetry of Clay [4] is a readable introduction. For Donaldson-Thomas invariants I would refer you to [5]. For Pandharipande-Thomas invariants the papers [6], [7] and [8] are best references.

[3] R. Pandharipande and R. P. Thomas. 13/2 ways of counting curves. *ArXiv e-prints*, November 2011

[4] Kentaro Hori, Sheldon Katz, Albrecht Klemm, Rahul Pandharipande, Richard Thomas, Cumrun Vafa, Ravi Vakil, and Eric Zaslow. *Mirror symmetry*, volume 1 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2003. With a preface by Vafa

[5] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. Gromov-Witten theory and Donaldson-Thomas theory, I. *ArXiv Mathematics e-prints*, December 2003

[6] R. Pandharipande and R. P. Thomas. Curve counting via stable pairs in the derived category. *Inventiones Mathematicae*, 178:407–447, May 2009

[7] R. Pandharipande and R. P. Thomas. The 3-fold vertex via stable pairs. *ArXiv e-prints*, September 2007

Chapter 1

Gromov-Witten Theory

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In a family of curves in a projective manifold X, a smooth curve can degenerate to a singular curve. For example the smooth conic $xy + z^2 = 0$ in $\mathbb{CP}^2$, we have $xy = 0$ which are a pair of lines $\mathbb{P}^1 \cup_{pt} \mathbb{P}^1$ which topologically looks like the wedge of two spheres. This is called a nodal singularity. Another deformation of this curves is $z^2 = 0$ which is a doubled line.

A smooth cubic in $\mathbb{P}^2$ is topologically a torus which can deform to a nodal cubic (i.e. locally $xy = 0$ around the singularity) which looks topologically like a $\mathbb{P}^1$ with two points of it glued together. The affine equation $x^2 = y^3$ in $\mathbb{P}^2$ is another type of singularity and there exists a degree one map from a smooth genus 0 curve to it.

Different moduli spaces handle these degenerations differently and lead to very different curve counting theories. In Gromov-Witten theory we consider curves as being given by maps $f : C \to X$. We require that the domain curve has at worst nodal singularities but we allow the map to not necessarily be an embedding.

1.1 *Moduli of stable maps*

Definition 1.1. A *stable map* to X of genus g and class $\beta \in H_2(X, \mathbb{Z})$ is a map $f : C \to X$ where C is a curve of arithmetic genus g with at worst nodal singularities, $f_*[C] = \beta \in H_2(X, \mathbb{Z})$ and such that

$$\mathrm{Aut}(f) := \{\varphi : C \xrightarrow{\cong} C, f \circ \varphi = f\}$$

is finite.

Definition 1.2. Two stable maps $f : C \to X$ and $f' : C' \to X$ are equivalent if there exists $\varphi : C \xrightarrow{\cong} C'$ such that the following diagram commutes

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\varphi"'] & X \\ C' \arrow[ur, "f'"'] & \end{tikzcd}$$

The reason for this definition is really the following theorem.

Theorem 1.3. *The moduli space of stable maps of fixed genus and degree denoted $\overline{\mathcal{M}}_g(X,\beta)$ is a projective Deligne-Mumford stack.*

In particular, this means that there is projective variety whose points are in bijective correspondence with isomorphism classes of stable maps.

Example 1.4. Consider $\overline{\mathcal{M}}_0(\mathbb{P}^2, 2H)$ where H is the class of a line $\mathbb{P}^1 \subseteq \mathbb{P}^2$. There is a map $\overline{\mathcal{M}}_0(\mathbb{P}^2, 2H) \to \mathbb{P}^5$ illustrating the fact that the conics in $\mathbb{P}^2$ form a linear system of dimension 5.

The nodal curve will be associated to an embedding of $\mathbb{P}^1 \cup_{pt} \mathbb{P}^1$ to the nodal curve. And this is unique up to isomorphism as shown in figure 1.1.

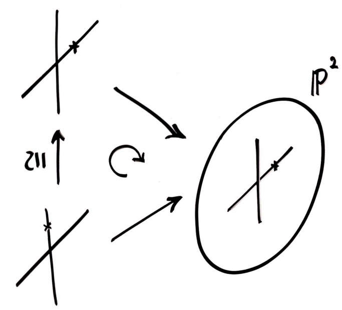

Figure 1.1: Embedding of the nodal curve is contributes to a unique map

The fiber of map over a double line $2H \subseteq \mathbb{P}^5$ is a copy of $\mathrm{Sym}^2 \mathbb{P}^1 = \mathbb{P}^1 \times \mathbb{P}^1/\mathbb{Z}/2 \cong \mathbb{P}^2$. When the two branch points come close to join together we get a map $\mathbb{P}^1 \cup_{pt} \mathbb{P}^1 \to \mathbb{P}^1$ (figure 1.2).

The locus of double lines in the linear system of conics is hence a copy of $\mathbb{P}^2$ sitting in $\mathbb{P}^5$ is the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.

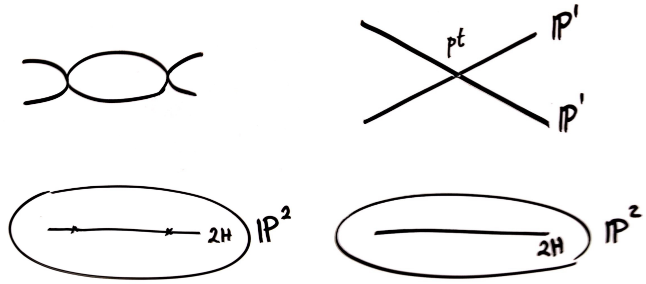

Figure 1.2: Stable mappings to double line

So $\overline{\mathcal{M}}_0(\mathbb{P}^2, 2[H]) \cong \mathrm{Bl}_{\mathbb{P}^2 \subseteq \mathbb{P}^5} \mathbb{P}^5$ and the exceptional divisor here are points with $\mathbb{Z}/2$ automorphisms.

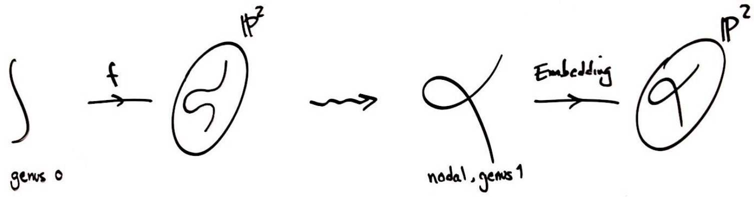

Figure 1.3: Nodal genus 1 curve

Example 1.5. What happens in $\overline{\mathcal{M}}_1(\mathbb{P}^2, 3[H])$? We have embeddings of genus 1 curves which can deform to nodal ones as well (figure 1.3). 1.7.

The limit of these are singularities locally looking like $x^2 = y^3$ (figure 1.4). But even more interesting phenomena can happen as shown in figures 1.5, 1.6 and

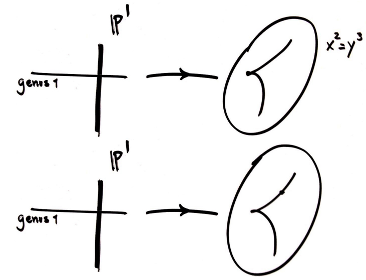

Figure 1.4: Some stable maps to the singularity $x^2 = y^3$

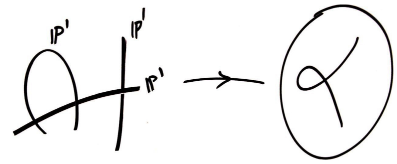

Figure 1.5: Some possible maps with genus one

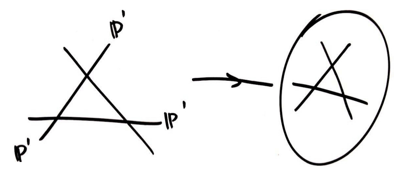

Figure 1.6: More possible maps with genus one

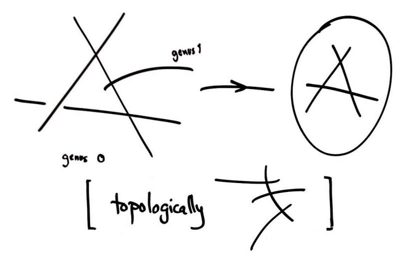

Figure 1.7: Even more possible maps with arithmetic genus one

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Theorem 1.6. *The virtual (or expected) dimension of $\overline{\mathcal{M}}_g(X, \beta)$ is*

$$\operatorname{vdim}(\overline{\mathcal{M}}_g(X, \beta)) = -K_X.\beta + (\dim_{\mathbb{C}} X - 3)(1 - g).$$

Here note as well that $-K_X.\beta = c_1(T_X)[\beta]$.

Example 1.7. If $\dim X = 3$ then the virtual dimension is independent of the generous. If X is a Calabi-Yau threefold then the virtual dimension is zero regardless of genus and homology class β.

Example 1.8. In $\mathbb{P}^2$ we have

$$\operatorname{vdim}(\mathcal{M}_g(\mathbb{P}^2, d[H])) = -\mathcal{O}(-3).d[H] + g - 1 = 3d + g - 1.$$

A short exercise is to show that this formula matches naive count coming from dimension of linear system $|\mathcal{O}(d)|$ and genus formula.

Suppose that $C \subset X$ is a smooth curve of genus g and degree β with normal bundle $N_{C/X} \to X$. A deformation of C can be thought of as a section of $N_{C/X}$. The long exact sequence associated to

$$0 \to T_C \to T_X|_C \to N_{C/X} \to 0$$

where $T_X|_C = f^* T_X$, is

$$\begin{aligned} 0 &\to H^0(C, T_C) \to H^0(C, T_X) \to H^0(C, N) \\ &\to H^1(C, T_C) \to H^1(C, T_X) \to H^1(C, N) \to 0. \end{aligned}$$

Here $H^0(C, N) = \operatorname{Def}(f : C \to X)$ and $H^1(C, T_C) = \operatorname{Def}(C)$ is the space of deformations of C. The group $H^1(C, T_X)$ is the space of obstructions $\operatorname{Ob}(f)$ and $H^1(C, N) = \operatorname{Ob}(f : C \to X)$. Also $H^0(C, T_C) = \operatorname{Aut}(C)$ and $H^0(C, T_X) = \operatorname{Def}(f)$. So the actual dimension of $T_{f:C\to X}\overline{\mathcal{M}}_g(X, \beta) = \dim H^0(C, N)$. And the expected dimension is

$$\begin{aligned} h^0(C, N) - h^1(C, N) &= \deg(N) + \dim N(1 - g) \\ &= -K_X.\beta + 2g - 2g + (\dim X - 1)(1 - g) \\ &= -K_X.\beta + (\dim X - 3)(1 - g) \end{aligned}$$

This calculation is valid when C is smooth. But of course in general C can be nodal and $f : C \to X$ does not have to be an embedding either.

Remark 1.9. If M is a variety, the Zariski tangent space T_pM of M at p can be given by the set of maps $\operatorname{Spec}(\mathbb{C}[x]/(x^2)) \to M$ that send the closed point of the domain to p. Hence $T_{[f:C\to X]}M$ is the space of families of maps over $\operatorname{Spec}\mathbb{C}[x]/(x^2)$. Calculating this space often turns out to have a homological method and is the topic of discussion in deformation theory.

A way to view this dimension formula is by looking at the moduli space of all nodal curves M_g which is smooth and of dimension $3g-3$ by Riemann-Roch and look at the fibbers of the map

$$p : \overline{\mathcal{M}}_g(X,\beta) \to M_g$$

and observe that

$$\text{vdim} = \dim M_g + \text{vdim}(\text{fiber of } p \text{ at } [C]).$$

The latter is again by Riemann-Roch just

$$\begin{aligned} h^0(c, f^*T_X) - h^0(C, f^*T_X) &= c_1(f^*T_X).[C] + \dim X.(1-g) \\ &= c_1(T_X).f_X[C] + \dim X(1-g) \\ &= -K_X.\beta + \dim X(1-g). \end{aligned}$$

$$\overline{M}_g(pt,0) = \overline{M}_g = \{C : \text{possibly nodal of genus } g \text{ }, |\operatorname{Aut}(C)| < \infty\}.$$

This is a smooth compact orbital of dimension $3g-3$. The map $\overline{\mathcal{M}}_g(X,\beta) \to \overline{M}_g$ sending $[f : C \to X]$ to C_{stab} where we take C and contract components with automorphisms.

Concretely what does $|\operatorname{Aut}(f : C \to X)| < \infty$ mean? It means that $f : C \to X$ is stable if every collapsing components C_i of genus zero has at least 3 nodes (and $\overline{M}_1(X,0)$ is empty!).

So for instance for $\overline{\mathcal{M}}_2(\mathbb{P}^2,[H]) \to \overline{M}_2$ we have the situation depicted in figure 1.9.

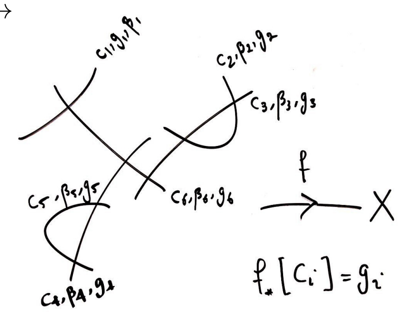

Figure 1.8: Characterization of stable maps

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1.2 *Marked points*

We define a new type of moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ via

$$\overline{\mathcal{M}}_{g,n}(X,\beta) = \left\{ f : (C, x_1, \cdots, x_n) \to X, \begin{array}{r} \text{(possibly nodal) genus } g \text{ curve} \\ \text{with } n \text{ non-singular distinct points} \\ x_1, \cdots, x_n \in C \text{ and } |\operatorname{Aut}(f)| < \infty \end{array} \right\}. \tag{1.1}$$

Stability then is the condition of every genus zero collapsing component has 3 or more points which are nodes or marked (and by definition $\overline{\mathcal{M}}_{1,0}(X,0) = \emptyset$). As a notation $\overline{\mathcal{M}}_{g,n} := \overline{\mathcal{M}}_{g,n}(pt,0)$.

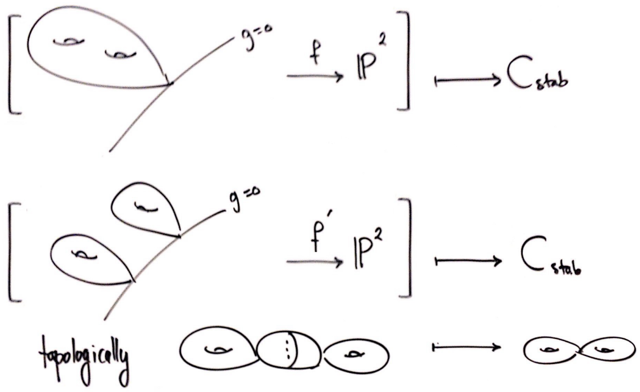
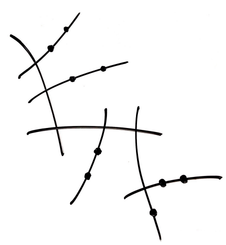


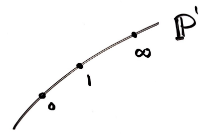
Figure 1.9: Elements of same fiber

Deligne-Mumford moduli space of stable curves are smooth orbifolds of dimension $3g-3+n$, and are define whenever $2g+n \geq 3$.

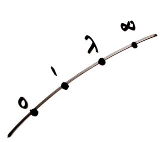
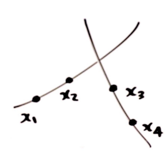
Example 1.10. $\overline{\mathcal{M}}_{0,n}$ is a manifold of dimension $n-3$.

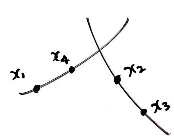
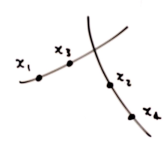
$\overline{\mathcal{M}}_{0,n} = \Bigg\{$  : trees of $\mathbb{P}^1$'s with 3 marked or nodal points on each component $\Bigg\}$.

Example 1.11. $\overline{\mathcal{M}}_{0,n}$ is a manifold of dimension $n-3$. With only three marked points the moduli space is a single point containing the projective line with three marked points on it:

$\overline{\mathcal{M}}_{0,3} = \Bigg\{$  $\Bigg\}$

In case of 4 marked points we have a strata of $\mathbb{P}^1 \setminus \{0,1,\infty\}$ and three points in the limits

$\overline{\mathcal{M}}_{0,4} = \Bigg\{$  $\Bigg\} \cup \Bigg\{$  $\Bigg\}$

$\cup \Bigg\{$  $\Bigg\} \cup \Bigg\{$  $\Bigg\}$.

which turns out to be $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$.

We have a forgetful map

$$\overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}$$

mapping $f : (C, x_1, \cdots, x_n) \to X$ to $(C, x_1, \cdots, x_n)_{stab}$ i.e. the curve with marked points as before except with non-stable components having been contracted. In particular relations in the cohomology of $\overline{\mathcal{M}}_{g,n}$ pullback to relations on $\overline{\mathcal{M}}_{g,n}(X, \beta)$. The results are (universal) relations among Gromov-Witten invariants.

1.3 Definition of Gromov-Witten invariants

The moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ also has maps

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$$

given by mapping $f : (C, x_1, \cdots, x_n) \to X$ to $f(x_i)$.

Suppose for instance that we want to count the number of curves passing through two points p and q:

$$\mathrm{ev}_1^{-1}(p) \cap \mathrm{ev}_2^{-1}(q) \subseteq \overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, H)$$

is the intersection of loci of maps where x_1 maps to p and x_2 maps to q. More generally if $A_1, \cdots, A_n \subset X$ are fixed submanifolds and

$$\mathrm{ev}_1^{-1}(A_1) \cap \cdots \cap \mathrm{ev}_n^{-1}(A_n) \subseteq \overline{\mathcal{M}}_{g,n}(X, \beta)$$

is finite then the cardinality of this set is the number of genus g curves of degree β passing through $A_1, \cdots, A_n$.

Dually we define

$$N_{g,\beta}^{GW}(A_1, \cdots, A_n) = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]} \mathrm{ev}_1^*(PD(A_1)) \cup \cdots \cup \mathrm{ev}_n^*(PD(A_n)). \tag{1.2}$$

But the fundamental class $[\overline{\mathcal{M}}_{g,n}(X, \beta)]$ and integration exist if $\overline{\mathcal{M}}_{g,n}(X, \beta)$ was smooth. In this case the above is an element of

$$H_{2\dim \overline{\mathcal{M}}_{g,n}(X,\beta)}(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Z}).$$

To make definition 1.2 work in general we need something that plays the role of the fundamental class in the smooth case.

Theorem 1.12. *There exists a class* $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir} \in H_*(X, \mathbb{Q})$ *the* virtual fundamental class *of degree* $2\,\mathrm{vdim}\, \overline{\mathcal{M}}_{g,n}(X, \beta)$.

Suppose we are given $E \to Y$ a vector bundle of rank r on a projective manifold of dimension d. We define a variety M by $M = s^{-1}(0) \subset Y$. The M has the expected dimension $d - r$ if s is transverse to the zero section. In this case M is a manifold of dimension $d-r$. If the fundamental class $[M]$ satisfies

$$i_*[M] = \underbrace{PD(\underbrace{c_r(E)}_{\in H^{2r}})}_{\in H_{2d-2r}} \in H_{2(d-r)}(Y, \mathbb{Z}).$$

If s is not transverse, then M can be singular and/or larger than expected dimension but there is a class $[M]^{vir} \in H_{2(d-r)}(M)$ and satisfying $i_*[M]^{vir} = PD(c_r(E))$.

Note that $[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}$ has $\mathbb{Q}$ coefficients because the fundamental class of an orbifold is only define over $\mathbb{Q}$: so $N^{GW}_{g,\beta}(A_1, \cdots, A_n) \in \mathbb{Q}$.

The number $N^{GW}_{g,\beta}(A_1, \cdots, A_n)$ is zero unless

$$-K_X.\beta + (\dim X - 3)(1-g) + n = \operatorname{codim}_{\mathbb{C}} A_1 + \cdots + \operatorname{codim}_{\mathbb{C}} A_n.$$

Each codim A_i cycle in X imposes codim $A_i - 1$ conditions on curves in X.

Example 1.13. If $\operatorname{codim} A = 1$, meaning A is a divisor, then asking that a curves passes through A imposes no conditions. (The number of intersections is determined cohomologically.) Asking a curve to pass through a point on a surface imposes one condition.

Example 1.14. If X is a Calabi-Yau threefold, the we need no conditions

$$N^{GW}_{g,\beta} := \int_{[\overline{\mathcal{M}}_{g,0}(X,\beta)]^{vir}} 1 \in \mathbb{Q}.$$

Gromov-Witten invariants are the closest to their enumerative interpretation when $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is smooth and of the expected dimension. Best set of examples is $g=0$ and $X = \mathbb{P}^n$. In this case $H^1(C, f^*T\mathbb{P}^n) = 0$ for all $(f : C \to X) \in \overline{\mathcal{M}}_{0,0}(\mathbb{P}^n, d[\ell])$. In other words, the obstructions vanish!

If $X = \mathbb{P}^2$ then $\dim \overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, d[\ell]) = 3d-1$

$$N_d = N^{GW}_{0,d[\ell]}(\overbrace{pt, \cdots, pt}^{3d-1}) = \begin{array}{l}\text{number of rational curves of degree } d\\ \text{passing through } 3d-1 \text{ points.}\end{array}$$

Theorem 1.15 (Kontsevich)**.** *For $d > 1$ we have*

$$N_d = \sum_{\substack{d_1+d_2=d\\ d_1,d_2>0}} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)$$

He proved this using *Quantum Cohomology* of $\mathbb{P}^2$. This structure is a deformation of the cup product on $H^*(\mathbb{P}^2)$ built from genus zero Gromov-Witten invariants.

The above formula follows from the associativity of the quantum product, which come from the relations on $\overline{\mathcal{M}}_{0,4}$ that the objects of figure 1.3 are homologous. We will discuss this in more details this week.

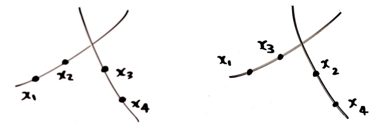


Figure 1.10: Homologous pointed stable maps

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1.4 Quantum Cohomology

This section is a digression of the mainstream of the course. It is not so much relevant to Calabi-Yau threefolds but interesting to get familiar with. As we said last time Quantum Cohomology is a deformation of the cup product on $H^*(X)$ depending on parameters built from genus zero Gromov-Witten invariants of X. The *quantum product* will be associative. Associativity arises from a non-trivial relation among the invariants. It is inherited from the trivial relation $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$.

First we put all invariants together in a generating function called *Gromov-Witten potential*. As a piece of notation we set

$$\langle \alpha_1, \cdots, \alpha_n \rangle_{0,\beta}^X = N_{0,\beta}^{GW}(\alpha_1, \cdots, \alpha_n)$$

for all $\beta \in H_2(X)$ and $\alpha_i \in H^*(X)$.

For convenience we assume that the cohomology of X is concentrated in even degrees: $H^*(X) = H^{even}(X)$. Then $\langle \cdot, \cdots, \cdot \rangle_{0,\beta}^X$ will be multi-linear and symmetric. So we may use monomial notation.

Example 1.16. The number of rational curves of degree d in $\mathbb{P}^2$ passing through $3d-1$ points, N_d, is given via

$$N_d = \langle \overbrace{pt, \cdots, pt}^{3d-1} \rangle_{0,d}^{\mathbb{P}^2} = \langle pt^{3d-1} \rangle_{0,d}^{\mathbb{P}^2}.$$

Let $\overbrace{T_0}^{1}, \overbrace{T_1, \cdots, T_p}^{H^2(X)}, \cdots, T_m \in H^*(X, \mathbb{Z})$ be a basis. Assume we have chosen $T_1, \cdots, T_p$ so that $\beta_i = T_i.\beta \geq 0$ for all curve classes β. Let

$$g_{ij} := \int_X T_i \cup T_j$$

and $\left(g^{ij}\right)$ be the inverse matrix. Let $t_0, \cdots, t_m$ be formal variables and let γ denote the summation $\gamma = t_0 T_0 + \cdots + t_m T_m$.

Definition 1.17. The genus zero Gromov-Witten potential is defined via

$$\begin{aligned} F &:= \sum_\beta \langle \exp(\gamma) \rangle_{0,\beta}^X q^\beta \qquad \text{where } q^\beta := q_1^{\beta_1} \cdots q_p^{\beta_p} \\ &= \sum_\beta \sum_{n=0}^\infty \frac{1}{n!} \langle (t_0 T_0 + \cdots + t_m T_m)^n \rangle_{0,\beta}^X q^\beta \\ &= \sum_\beta \sum_{n_0, \cdots, n_m \geq 0} \langle T_0^{n_0}, \cdots, T_m^{n_m} \rangle_{0,\beta}^X \frac{t_0^{n_0}}{n_0!} \cdots \frac{t_m^{n_m}}{n_m!} q^\beta \end{aligned}$$

which is an element of $\mathbb{Q}[\![q_1, \cdots, q_p, t_0, \cdots, t_m]\!]$ as a formal generating function.

Coefficients of the GW potential are all possible genus-zero Gromov-Witten invariants of X. In this generating function t_i's are tracking insertions and q_i's are tracking β.

Note 1.18. The partial derivatives of F are interesting:

$$\frac{\partial^\ell F}{\partial t_{i_1} \cdots \partial t_{i_\ell}} = \sum_\beta \langle \exp(\gamma) T_{i_1} \cdots T_{i_\ell} \rangle_{0,\beta} q^\beta$$
$$\left.\frac{\partial^\ell F}{\partial t_{i_1} \cdots \partial t_{i_\ell}}\right|_{t=0} = \sum_\beta \langle T_{i_1} \cdots T_{i_\ell} \rangle_{0,\beta} q^\beta$$

Question: What happens at $q = 0$, i.e. what are $\langle T_0^{n_0} \cdots T_m^{n_m} \rangle_{0,0}^X$?

$$\langle \gamma_1, \cdots, \gamma_n \rangle_{0,0} = \int_{[\overline{\mathcal{M}}_{0,n}(X,0)]^{vir}]} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n)$$

But note that $\overline{\mathcal{M}}_{0,n}(X, 0) = \overline{M}_{0,n} \times X$ with the mapping

$$(f : (C, x_1, \cdots, x_n) \to X) \mapsto ((C, x_1, \cdots, x_n), \mathrm{im}(f)).$$

and has dimension $-K_X.\beta + (\dim X - 3)(1 - g) + n = n - 3 + \dim X$ since $g = 0$ and $\beta = 0$. Hence this space is smooth of expected dimension so the virtual class is the fundamental class of the variety.

In this case $\mathrm{ev}_i : \overline{M}_{0,n} \to X$ is just the projection π_X, on the second factor. And we have,

$$\begin{aligned}\langle \gamma_1, \cdots, \gamma_n \rangle_{0,0}^X &= \int_{\overline{M}_{0,n} \times X} \underbrace{\pi_X^*(\gamma_1) \cdots \pi_X^*(\gamma_n)}_{3-n+\dim X} \\ &= \begin{cases} 0 & \text{if } n \neq 3 \\ \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3 & \text{if } n = 3 \end{cases}\end{aligned}$$

We conclude that

$$F(t, q)|_{q=0} = \frac{1}{6} \int_x (t_0 T_0 + \cdots + t_m T_m)^3.$$

The third partials $F_{\alpha\beta\gamma} := \frac{\partial^3 F}{\partial t_\alpha \partial t_\beta \partial t_\gamma}$ satisfy

$$F_{\alpha\beta\gamma}\big|_{q=0} = \int_X T_\alpha \cup T_\beta \cup T_\gamma.$$

And the Poincare pairing is just

$$F_{0\beta\gamma}\big|_{q=0} = g_{\beta\gamma}.$$

Note 1.19. Note that

$$T_\alpha \cup T_\beta = \sum_{\varepsilon, \varepsilon'} F_{\alpha\beta\varepsilon}\big|_{q=0} g^{\varepsilon\varepsilon'} T_{\varepsilon'}.$$

Definition 1.20. We define a product $*$ on $H^*(X) \otimes \mathbb{Q}[\![q_1, \cdots, q_p, t_0, \cdots, t_m]\!]$ by the formula

$$T_\alpha * T_\beta = \sum_{\varepsilon,\varepsilon'} F_{\alpha,\beta,\varepsilon} g^{\varepsilon\varepsilon'} T_{\varepsilon'}$$

This is obviously commutative and when $q = 0$ then the product is 0.

Theorem 1.21. *The product $*$ is an associative product.*

In other words since

$$\begin{aligned}(T_\alpha * T_\beta) * T_\gamma &= (F_{\alpha\beta\varepsilon} g^{\varepsilon\varepsilon'} T_{\varepsilon'}) * T_\gamma \\ &= F_{\alpha\beta\varepsilon} g^{\varepsilon\varepsilon'} F_{\varepsilon'\gamma\delta} g^{\delta\delta'} T_{\delta'} \\ (T_\beta * T_\gamma) * T_\alpha &= F_{\beta\gamma\varepsilon} g^{\varepsilon\varepsilon'} F_{\varepsilon'\alpha\delta} g^{\delta\delta'} T_{\delta'}\end{aligned}$$

which means that for all α, β, γ, and δ we have the WDVV equations

$$F_{\alpha,\beta,\varepsilon} g^{\varepsilon\varepsilon'} F_{\varepsilon'\gamma\delta} = F_{\beta\gamma\varepsilon} g^{\varepsilon\varepsilon'} F_{\varepsilon',\alpha,\delta}.$$

These are partial differential equations for F. Before we prove WDVV let's see what F looks like in some examples. First we look at a few simple relations:

<u>String equation</u>: Note that $\langle \gamma_1, \cdots, \gamma_n, T_0 \rangle_{0,\beta} = 0$ unless $\beta = 0, n = 2$. This is because

$$\langle \gamma_1, \cdots, \gamma_n, T_0 \rangle_\beta = \int_{[\overline{\mathcal{M}}_{0,n+1}(X,\beta)]^{vir}} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_n^*(\gamma_n) \cup \overbrace{\mathrm{ev}_{n+1}^*(T_0)}^{1}$$

The forgetful map

$$\overline{\mathcal{M}}_{0,n+1}(X,\beta) \to \overline{\mathcal{M}}_{0,n}(X,\beta)$$

exists as long as $\overline{\mathcal{M}}_{0,n}(X,\beta)$ is non-empty. So the integrand pulls back from $\overline{\mathcal{M}}_{0,n}$ which has smaller dimension!

So what this implies about our potential function is that

$$\begin{aligned}\frac{\partial F}{\partial t_0} &= \sum_\beta \langle \exp(\gamma) T_0 \rangle_\beta q^\beta = \frac{1}{2} \langle \gamma^2 T_0 \rangle_{0,0} \\ &= \frac{1}{2} \int_X \gamma^2 = \frac{1}{2} \sum_{i,j} t_i t_j g_{ij}\end{aligned}$$

$F_{0\alpha\beta} = g_{\alpha\beta}$ so $T_0 = \mathrm{id}$ for the product $*$.

<u>Divisor equation</u>: For all $i \in \{1, \cdots, p\}$ and $T_i \in H^2(X)$,

$$\langle \gamma^n, T_u \rangle_{0,\beta} = (\underbrace{T_i \cdot \beta}_{\beta_i}) \langle \gamma^n \rangle_{0,\beta}$$

unless $\beta = 0$ and $n = 2$.

$$\begin{aligned}\frac{\partial F}{\partial t_i} &= \sum_\beta \langle \exp(\gamma) T_i \rangle_\beta q^\beta \\ &= \beta_i \sum \langle \exp(\gamma) \rangle_\beta q^\beta + \frac{1}{2} \langle \gamma^2, T_i \rangle_{0,0}\end{aligned}$$

So

$$\boxed{\frac{\partial F}{\partial t_i} = q_i \frac{\partial F}{\partial q_i} - \frac{1}{2}\int_X \gamma^2 T_i} \tag{1.3}$$

known as the Divisor equation. This implies that

$$\left(\frac{\partial}{\partial t_i} - q_i \frac{\partial}{\partial q_i}\right) F_{\alpha\beta\gamma} = 0$$

expect for cubic terms in t when $q = 0$. The dependence of F on t_i and q_i is as a function of $q_i e^{t_i}$. We know are going to attempt to finding F in some cases.

Example 1.22. First consider the case of $X = \mathbb{P}^1$. Here $T_0 = 1$ and $T_1 = [pt]^\vee$. In $\beta = 0$ the only non-zero invariant is $\langle 1, 1, [pt]^\vee \rangle = 1$. In $\beta = d \neq 0$ we have

$$\langle T_1^\ell \rangle_d = d^\ell \langle . \rangle_d = \begin{cases} 0 & d \neq 1 \\ 1 & d = 1 \end{cases} .$$

And vdim $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, d[\mathbb{P}^1]) = -K_X\beta + (\dim X - 3)(1-g) = 2d - 2.$

We conclude that

$$F = \frac{1}{2} t_0^2 t_1 + q_1 \sum_\ell \frac{1}{\ell!} \langle T_1^\ell \rangle t_1^\ell = \frac{1}{2} t_0^2 t_1 + q_1 e^{t_1}.$$

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Example 1.23. Our second example is the case of $X = \mathbb{P}^2$. Recall the string equation says $\langle T_0, \gamma_1, \cdots, \gamma_n \rangle_{0,\beta} = 0$ unless $n = 2$ and $\beta = 0$ and that

$$\langle T_\alpha, T_\beta, T_\gamma \rangle_{0,0} = \int_X T_1 \alpha \cup T_\beta \cup T_\gamma.$$

As a basis $H^*(\mathbb{P}^2)$ take $T_0 = 1$, $T_1 = H^\vee$ where $H \subseteq \mathbb{P}^2$ is an embedded projective line and $T_2 = pt^\vee$. The paring matrix is

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

$$F = \frac{t_0 t_1^2}{2} + \frac{t_0^2 t_2}{2} + \sum_{d=1}^{\infty} \sum_{n_1, n_2 \geq 0} \langle pt^{n_2}, H^{n_1} \rangle_{0,d} q^d \frac{t_1^{n_!}}{n_1!} \frac{t_2^{n_2}}{n_2!}$$

Here $n_2 = 3d - 1$ and by divisor equation we can pull H^{n_1} out and get

$$\begin{aligned} F &= \frac{t_0 t_1^2}{2} + \frac{t_0^2 t_2}{2} + \sum_{d=1}^{\infty} q^d \sum_{n_1 \geq 0}^{\infty} d^{n_1} \underbrace{\langle pt^{3d-1} \rangle_{0,d}}_{N_d} \frac{t_1^{n_!}}{n_1!} \frac{t_2^{3d-1}}{(3d-1)!} \\ &= \frac{t_0 t_1^2}{2} + \frac{t_0^2 t_2}{2} + \sum_{d=1}^{\infty} q^d e^{dt_1} N_d \frac{t_2^{3d-1}}{(3d-1)!} \end{aligned}$$

Notice how appearance of $q^d e^{dt_1}$ is a common behavior is divisor equation. Now we look at WDVV equations

$$F_{\alpha\beta\varepsilon} g^{\varepsilon\varepsilon'} F_{\varepsilon'\beta\delta} = F_{\beta\gamma\varepsilon} g^{\varepsilon\varepsilon'} F_{\varepsilon'\alpha\delta}$$

And recalling what (g_{ij}) is here we get

$$\underbrace{F_{110}}_{1} F_{222} + F_{111} F_{122} + F_{112} \underbrace{F_{022}}_{0} = \underbrace{F_{120}}_{0} F_{212} + F_{121} F_{112} + F_{122} \underbrace{F_{012}}_{0}$$

simplifying to

$$F_{222} = F_{112}^2 - F_{111} F_{122}.$$

Taking partials of the Gromov-Witten potential now tells us

$$\begin{aligned}
F_{222} &= \sum_d (qe^{t_1})^d N_d \frac{t_2^{3d-4}}{(3d-4)!} \\
F_{112} &= \sum_d d^2 (qe^{t_1})^d N_d \frac{t_2^{3d-2}}{(3d-2)!} \\
F_{111} &= \sum_d d^3 (qe^{t_1})^d N_d \frac{t_2^{3d-1}}{(3d-1)!} \\
F_{122} &= \sum_d d (qe^{t_1})^d N_d \frac{t_2^{3d-3}}{(3d-3)!}.
\end{aligned}$$

Hence WDVV is

$$\sum_{d>1} N_d \frac{t^{3d-4}}{(3d-4)!} = \sum_{d_1,d_2>0} N_{d_1} N_{d_2} \left(\frac{d_1^2 d_2^2 t^{3(d_1+d_2)-4}}{(3d_1-2)!(3d_2-2)!} - \frac{d_1^3 d_2 t^{3(d_1+d_2)-4}}{(3d_1-1)!(3d_2-3)!} \right)$$

This means that N_d is $(3d-4)!(t^{3d-4}$ coefficient of $q)$ resulting Kontsevich's surprising result

$$\boxed{N_d = \sum_{\substack{d_1+d_2=d \\ d_1,d_2>0}} N_{d_1} N_{d_2} \left(d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right)} \tag{1.4}$$

(Kontsevich's Formula)

Let's finish this discussion with a sketch of the proof of WDVV equation in the case where $\overline{\mathcal{M}}_{0,n}(X,\beta)$ are smooth and of the expected dimension at least in the example of $X = \mathbb{P}^n$ (where we do not need a virtual class).

Consider the following integral

$$\begin{aligned}
(*)_{n,\alpha,\beta,\gamma,\delta} := \frac{1}{n!} \int_{[\overline{\mathcal{M}}_{0,n+4}(X,\beta)]} \rho^*(pt^\vee) \operatorname{ev}_1^*(T_\alpha) \operatorname{ev}_2^*(T_\beta) \operatorname{ev}_3^*(T_\gamma) \\
\operatorname{ev}_4^*(T_\delta) \operatorname{ev}_5^*(\gamma) \cdots \operatorname{ev}_{n+4}^*(\gamma)
\end{aligned}$$

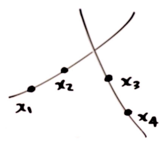
where

$$\begin{aligned}
\rho : \overline{\mathcal{M}}_{0,n+4}(X,\beta) &\to \overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1 \\
[f : (C, x_1, \cdots, x_{n+4} \to X] &\mapsto (C, x_1, \cdots, x_4)_{stab}
\end{aligned}$$

We are assuming everything is smooth and ρ is transverse so we may write

$$\rho^*(pt^\vee) = (\rho^{-1}(pt))^\vee.$$

We can choose the point pt to be any point in $\overline{\mathcal{M}}_{0,4}$:

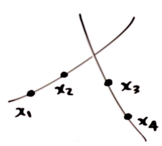
$$\Bigg\{$$


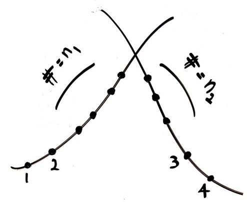
$$\Bigg\} \in \overline{\mathcal{M}}_{0,4}.$$

So now we try to understand the integral $*$ in these terms:

$$(*) = \frac{1}{n!}\int_{\rho^{-1}(pt)} \mathrm{ev}_1^*(T_\alpha) \cup \cdots \cup \mathrm{ev}_{n+4}^*(\gamma).$$

So

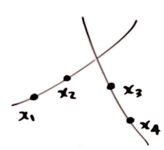
$$\rho^{-1}\left(\Bigg\{\right.$$


$$\left.\Bigg\}\right) = \Bigg\{ f :$$


$$\to X : \begin{matrix} n_1 + n_2 = n \\ \beta_1 + \beta_2 = \beta \end{matrix}\Bigg\}$$

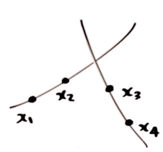
$$\subseteq \coprod_{\substack{n_1+n_2=n \\ \beta_1+\beta_2=\beta}} \overline{\mathcal{M}}_{0,n_1+3}(X,\beta_1) \times \overline{\mathcal{M}}_{0,n_2+3}(X,\beta_2)$$

where the disjoint union index n really ranges over all partitions $\{5,\cdots,n+4\} = A_1 \cup A_2$ where $|A_i| = n_i$. So

$$\rho^{-1}\left(\Bigg\{\right.$$


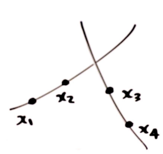
$$\left.\Bigg\}\right) = \left({}_L\,\mathrm{ev}_{n_1+3}^* \times \cup\, {}_R\,\mathrm{ev}_{n_2+3}^*\right)^{-1}(\Delta)$$

where $\Delta \subset X \times X$ is the diagonal, and dually

$$\rho^*\left(\Bigg\{\right.$$


$$\left.\Bigg\}^\vee\right) = \left({}_L\,\mathrm{ev}_{n_1+3}^* \times \cup\, {}_R\,\mathrm{ev}_{n_2+3}^*\right)^*(\Delta^\vee)$$

but in any manifold $\Delta^\vee = T_\varepsilon \otimes T^\varepsilon$ (suppressing the summation notation over ε hence

$$\rho^{-1}\left(\Bigg\{\right.$$


$$\left.\Bigg\}\right) = {}_L\,\mathrm{ev}_{n_1+3}^*(T_\varepsilon) \cup {}_R\,\mathrm{ev}_{n_2+3}^*(T^\varepsilon).$$

So the integral is

$$(*) = \frac{1}{n!} \sum_{\substack{n_1+n_2=n \\ \beta_1+\beta_2=\beta}} \binom{n}{n_1} \int_{[\overline{\mathcal{M}}_{0,n_1+3}(X,\beta_1)\times\overline{\mathcal{M}}_{0,n_2+3}(X,\beta_2)]} {}_L\operatorname{ev}^*_{n_1+3}(T_\varepsilon) \cup {}_R\operatorname{ev}^*_{n_2+3}(T^\varepsilon)$$
$$ {}_L\operatorname{ev}^*_1(T_\alpha) \cup {}_L\operatorname{ev}^*_2(T^\beta)\, {}_R\operatorname{ev}^*_1(T_\gamma) \cup {}_R\operatorname{ev}^*_2(T^\delta)$$
$$ {}_L\operatorname{ev}^*_3(\gamma)\cdots {}_L\operatorname{ev}^*_{n_1+2}(\gamma)\, {}_R\operatorname{ev}^*_3(\gamma)\cdots {}_R\operatorname{ev}^*_{n_2+2}(\gamma)$$
$$= \sum_{\substack{n_1+n_2=n \\ \beta_1+\beta_2=\beta}} \frac{1}{n_1!n_2!} \langle \gamma^{n_1} T_\alpha T_\beta T_\varepsilon \rangle_{0,\beta_1} \langle \gamma^{n_2} T_\gamma T_\delta T^\varepsilon \rangle_{0,\beta_2}$$

We conclude that

$$\sum_{n,\beta} (*) q^\beta = \sum_{\beta_1,\beta_2} \langle \exp(\gamma) T_\alpha T_\beta T_\varepsilon \rangle_{0,\beta_1} q^{\beta_1} \langle \exp(\gamma) T_\gamma T_\delta T_{\varepsilon'} g^{\varepsilon\varepsilon'} \rangle_{0,\beta_2} q^{\beta_2}$$
$$= F_{\alpha,\beta,\varepsilon} F_{\gamma\delta\varepsilon'} g^{\varepsilon\varepsilon'}$$

We therefore conclude that this is symmetric in α, β, γ and δ. This is very interesting, in fact the existence of a quantum product on the structure of any manifold is interesting!

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1.5 *Gromov-Witten invariants of Calabi-Yau manifolds*

We discussed the Gromov-Witten potential in previous lectures which was a big generating function with variables t_αs and q_is tracking insertions and track degree. We return now to Calabi-Yau threefolds in which case we recall that $\operatorname{vdim} \overline{\mathcal{M}}_g(X,\beta) = 0$. All information comes at $t = 0$.

Definition 1.24. The genus g Gromov-Witten potential of a Calabi-Yau threefold X is [1]

$$F^g = \sum_\beta N^{GW}_{g,\beta} v^\beta.$$

[1] The notation changed from q to v through the history of these invariants as use of q became more prominent in Donaldson-Thomas theory.

This series lives in $\mathbb{Q}[\![v_1, \cdots, v_p]\!]$ where $v^\beta = v_1^{\beta_1} \cdots v_p^{\beta_p}$. Usually this includes $\beta = 0$ when $g \geq 0$ but sometimes this sum is written over $\beta \neq 0$.

Definition 1.25. The all-genus potential function is the formal Laurent series

$$F = \sum_g F^g \lambda^{2g-2}$$

in λ. The variable λ is known as *string coupling* constant.

One more piece of definition is

Definition 1.26. The Gromov-Witten partition function is defined as

$$Z = \exp(F) \qquad \text{and} \qquad Z^0 = \exp(F') = \frac{\exp(F)}{\exp(F|_{v=0})} = \frac{Z}{Z|_{v=0}}.$$

Homework 1.27. Show that if we write

$$Z = \sum_{\chi} \sum_{\beta} N^{\bullet}_{\chi,\beta} \lambda^{-2\chi} v^{\beta}.$$

Then $N^{\bullet}_{\chi,\beta}$ can be interpreted as *disconnected* Gromov-Witten invariants:

$$N^{\bullet}_{\chi,\beta} = \int_{[\overline{\mathcal{M}}^{\bullet}_{\chi}(X,\beta)]^{vir}} 1$$

where

$$\overline{\mathcal{M}}^{\bullet}_{\chi}(X,\beta) = \left\{ f : C \to X, \begin{array}{c} \text{possibly disconnected and/or nodal} \\ \text{curves such that } f_*[C] = \beta \text{ and} \\ \chi(\mathcal{O}_C) = \chi \text{ and } |\operatorname{Aut}(f)| < \infty \end{array} \right\}.$$

where $\chi(\mathcal{O}_C)$ is the number of components minus the sum of genera of the components. You may assume the following reasonable behavior for the virtual funcdamental class, that

1. $[M_1 \sqcup M_2]^{vir} = [M_1]^{vir} + [M_2]^{vir}$
2. $\deg[M_1 \times M_2]^{vir} = \deg[M_1]^{vir}. \deg[M_2]^{vir}$
3. $\deg[M/G]^{vir} = \frac{1}{|G|} \deg[M]^{vir}$ where G is a finite group.

Unlike genus-zero Gromov-Witten theory of $\mathbb{P}^2$ the relationship between enumerative geometry (i.e. curve counting) and Gromov-Witten invariants is more subtle.

Example 1.28. $X^3_{(5)} \subseteq \mathbb{P}^4$ generic quintic threefold. This has 2875 lines on it. They are all homologous and they generate $H_2(X,\mathbb{Z})$. They all have normal bundle $N_{\mathbb{P}^1/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Note that since we are in a Calabi-Yau manifold, the degree of the normal bundle (here 2) is the same as the canonical bundle of $\mathbb{P}^1$ as is the case here. Every degree 2 curve in X is a smooth plane conic. There are 609250 of them. There are no curves of genus strictly greater than zero in degrees 1 and 2.

So what are the corresponding Gromov-Witten invariants. Let us write d for, $d[\ell]$, i.e. d-times class of a line. And then the invariants we are looking for are denoted by $N_{g,d}$.

Every stable map of degree 1, genus zero is an isomorphism on to its image:

$$N_{0,1} = 2875$$

stable maps of degree 2, and genus zero are either isomorphisms onto one of the conics or a degree two multiple cover of a line. The second factor is the contributions of multiple covers of lines.

$$N_{0,2} = 609250 + {}^{1}/{}_{8}2875$$

In fact

$$\overline{\mathcal{M}}_0(X,2) = \bigsqcup_{609250} \overline{\mathcal{M}}_0(\mathbb{P}^1, [\mathbb{P}^1]) \bigsqcup_{2875} \overline{\mathcal{M}}_{0(\mathbb{P}^1, 2[\mathbb{P}^1])}$$

But notice that the second types of strata have dimension two which is bigger than the expected dimension. Hence it is a complicated problem to understand how to account for its contribution.

In higher genera, for instance we have

$$N_{1,1} = {}^{1}/_{12}(2875)$$

because

$$\overline{\mathcal{M}}_1(X,1) = \bigsqcup_{2875} \underbrace{\overline{\mathcal{M}}_1(\mathbb{P}^1,[\mathbb{P}^1])}_{\mathbb{P}^1\times\overline{\mathcal{M}}_{1,1}}.$$

1.6 *Multiple cover formula–local* $\mathbb{P}^1$

The fundamental problem here is what is the contribution of an isolated curve of genus g and degree β in X to $N_{g+h,d\beta}$? Notice that Gromov-Witten invariants are not just well-defined for algebraic varieties but also symplectic manifolds. And in this general framework these embedded curves are isolate.

For example if $\mathbb{P}^1 \subseteq X$ with $N_{\mathbb{P}^1/X} \cong \mathcal{O}(-1)\oplus\mathcal{O}(-1)$ then this $\mathbb{P}^1$ is super-rigid in X, meaning there are no deformations (even infinitesimal) of maps to $\mathbb{P}^1$ which deform off of $\mathbb{P}^1$. In other words $\overline{\mathcal{M}}_g(\mathbb{P}^1,d[\mathbb{P}^1])$ lives as a component of the moduli space:

$$\overline{\mathcal{M}}_g(\mathbb{P}^1,d[\mathbb{P}^1]) \subseteq \overline{\mathcal{M}}_g(X,d[\mathbb{P}^1])$$

and we can restrict $[\overline{\mathcal{M}}_g(X,d[\mathbb{P}^1])]^{vir}$ to this component and we get a well-defined contribution

$$N^{GW}_{g,d}(\text{ local }\mathbb{P}^1) = \int_{[\overline{\mathcal{M}}_g(X,d[\mathbb{P}^1])]^{vir}|_{\overline{\mathcal{M}}_g(\mathbb{P}^1,d[\mathbb{P}^1])}} 1$$

This is in other words, the universal multiple cover (or degenerate map) contributions.

The conifold singularity is $\operatorname{Spec}\mathbb{C}[x,y,z,w]/(xy-zw)$, which is a three dimensional variety with a singularity at the origin. This has a resolution (called conifold resolution)

$$X \cong \operatorname{tor}(\mathcal{O}(-1)\oplus\mathcal{O}(-1)\to\mathbb{P}^1)$$

This is a famous model in Gromov-Witten theory and string theory and is sometimes called *the local-*$\mathbb{P}^1$.

We want to find the potential and partition function of this space:

$$N^{GW}_{g,d}(X) = \int_{[\overline{\mathcal{M}}_g(X,d[\mathbb{P}^1])]^{vir}} 1 = \int_{[\overline{\mathcal{M}}_g(X,d[\mathbb{P}^1])]^{vir}} \operatorname{eu}(\mathrm{Ob})$$

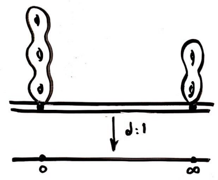
the integrand is the Euler class of the obstruction sheaf $\mathrm{Ob}\to\overline{\mathcal{M}}_g(\mathbb{P}^1,d[\mathbb{P}^1])$. In particular,

$$\mathrm{Ob}|_{f:C\to\mathbb{P}^1} = H^1(C,f^*N_{\mathbb{P}^1/X}) = H^1(C,f^*(\mathcal{O}(-1)\oplus\mathcal{O}(-1)))$$

when C is smooth.

This is computed via (virtual) localization. This technique uses $\mathbb{C}^*$ action on $\overline{\mathcal{M}_g}(\mathbb{P}^1, d[\mathbb{P}^1])$ induced by the action of $\mathbb{C}^*$ on $\mathbb{P}^1$. This reduces the integration to an integral over $\overline{\mathcal{M}_g}(\mathbb{P}^1, d[\mathbb{P}^1])^{\mathbb{C}^*}$. And this moduli space can be explicitly described in terms of $\overline{\mathcal{M}}_{g_1,n} \times \overline{\mathcal{M}}_{g_2,n}$.

$$\overline{\mathcal{M}_g}(\mathbb{P}^1, d[\mathbb{P}^1])^{\mathbb{C}^*} = \left\{ \quad : g_1 + g_2 = g \right\}$$



We will learn how to do localization and will be able to calculate this integral. We will see that localization will reduce the problem to an integral over $\coprod_{g_1+g_2=g} \overline{\mathcal{M}}_{g_1,n} \times \overline{\mathcal{M}}_{g_2,n}$. Eventually we will see that

$$N_{g,d}(\text{ local } \mathbb{P}^1) = d^{2g-3} \sum_{g_1+g_2=g} b_{g_1} b_{g_2}$$

by Bernoulli numbers that are defined for instance via

$$\sum_{g=0}^{\infty} b_g t^{2g} = \left(\frac{\sin(t/2)}{t/2} \right)^{-1}$$

The potential function will therefore be

$$\begin{aligned} F = \sum_{g,d:d\neq 0} N_{g,d} \lambda v^d &= \sum_{g_1,g_2 \geq 0, d>0} d b_{g_1} b_{g_2} \lambda^{2g_1+2g_2-2} v^d \\ &= \sum_{d>0} v^d d^{-3} \lambda^{-2} \left(\sum_{g_1} b_{g_1} d^{2g_1} \lambda^{2g_1} \right) \left(\sum_{g_2} b_{g_2} d^{2g_2} \lambda^{2g_2} \right) \\ &= \sum_{d>0} v^d d^{-3} \lambda^{-2} \left(\frac{\sin(d\lambda/2)}{d\lambda/2} \right)^{-2} \\ &= \sum_{d>0} \frac{v^d}{d} \left(2 \sin(d\lambda/2) \right)^{-2}. \end{aligned}$$

In genus zero, F^0 is λ^{-2}-term of the above which is

$$\boxed{F^0 = \sum_{d>0} \frac{v^d}{d^3}} \tag{1.5}$$

(Aspinwall-Morrison Formula)

This $\frac{1}{d^3}$ is the multiple cover contribution in genus zero.

Jan 29

1.7 *Multiple cover formula*

Recall that we have been discussing contributions from multiple covers and degenerate contributions. The general idea, is to simplify

Gromov-Witten theory of Calabi-Yau threefolds by systematically removing multiple cover contributions.

Last time we began to execute this idea if $\mathbb{P}^1 \subseteq X$ with $N_{\mathbb{P}^1/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. We observed that the contribution of this $\mathbb{P}^1$ to $N^{GW}_{g,d[\mathbb{P}^1]}$ is well-defined because

$$\overline{\mathcal{M}}_g(\mathbb{P}^1, d[\mathbb{P}^1]) \subset \overline{\mathcal{M}}_g(X, d[\mathbb{P}^1])$$

as an isolated connectedness component. In other words not just that these maps from $\mathbb{P}^1$ do not deform in families but they do not deform infinitesimally either. In general we have

Definition 1.29. A curve $C \subset X$ is super-rigid, if

$$\overline{\mathcal{M}}_g(C, d[C]) \subseteq \overline{\mathcal{M}}_g(X, d[C])$$

as a scheme-theoretic connected component. Then we define

$$N^{GW}_{g,d}(\text{ local } C) = \int_{[\overline{\mathcal{M}}_g(X,d[C])]^{vir}|_{\overline{\mathcal{M}}_g(C,d[C])}} 1.$$

Note that super-rigidity is easily seen to be satisfied in the symplectic world. But is a more subtle condition here. However we still expect it to be a reasonable criterion.

Example 1.30. Last time we derived that $N^{GW}_{g,d}(\text{ local } \mathbb{P}^1)$ has potential function

$$F' = \sum_{d>0,g} N_{g,d} \lambda^{2g-2} v^d = \sum_{d>0} \frac{v^d}{d} \left(2 \sin \left(\frac{d\lambda}{2} \right) \right)^{-2}.$$

Hence $F^0 = \sum_d \frac{1}{d^3} v^d$ which is the simplest of multiple-cover formulae.

Note that although this looks like a nice formula, it is actually pretty complicated; every $\mathbb{P}^1$ is contributing in every degree, and in every genus. So these types of formulae beg for some unraveling.

Example 1.31. Suppose $E \subset X$ is a smooth elliptic curve with $N_{E/X} \cong L \oplus L^{-1}$ where L is a degree zero line bundle. And since we are talking about elliptic curves we have $L \in \mathrm{Pic}^0(E)$. A line bundle is non-trivial is it does not have any non-zero sections. But if we want it to not have any multi-sections then it is *non-torsion*, i.e. $L^{\otimes n} \ncong \mathcal{O}_X$ for any n. Then $E \subseteq X$ is super-rigid so we get

$$N^{GW}_{g,d}(\text{ local } E) = \int_{[\overline{\mathcal{M}}_g(X,d[E])]^{vir}|_{\overline{\mathcal{M}}_g(X,d[E])}} 1.$$

And one can use degeneration techniques to prove the following

Theorem 1.32. *For genera other than 1, $g \neq 1$ we have for an elliptic curve*

$$N^{GW}_{g,d}(\textit{local } E) = 0.$$

Homework 1.33. Use the covering space theory to show that there are $\sigma(d) = \sum_{k|d} k$ (connected) covering spaces of degree d over E and each is a normal covering space with cardinality of the group of deck transformations being d

So what does this tell a Gromov-Witten theorist? To compute $N^{GW}_{1,d}($ local $E)$ we need to study $\overline{\mathcal{M}}_1(E, d[E])$. By Riemann-Hurwitz formula $f : E \to E$ is unramified with smooth domain. Thus

$$\overline{\mathcal{M}}_1(E, d[E]) = \sigma(d) \text{ isolated points}$$

and each of these points is an orbifold point of the form $[*/G]$ with $|G| = d$. And this implies that the fundamental class is just the sum of these points devided by order of the group. So eventually we conclude that these invariants are

$$N^{GW}_{g,d}(\text{ local } E) = \begin{cases} 0 & g \neq 1 \\ \frac{\sigma(d)}{d} & g = 1 \end{cases}.$$

So we have

$$\begin{aligned} F' &= \sum_{d>0} N_{1,d} v^d = \sum_{d>0} \frac{1}{d} \sigma(d) v^d \\ &= \sum_{d=1}^{\infty} \frac{1}{d} \sum_{k|d} k v^d = \sum_{k,m \geq 1} \frac{k}{km} v^{km} \\ &= \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} (v^k)^m = \sum_{k=1}^{\infty} - \log(1 - v^k) \end{aligned}$$

So we have

$$F' = \log(\prod_{k=1}^{\infty} \frac{1}{(1 - v^k)}).$$

And

$$Z' = \exp(F') = \prod_{k=1}^{\infty} \frac{1}{(1 - v^k)} = \sum_{n=0}^{\infty} p(n) v^n.$$

where $p(n)$ is the number of partitions of n, i.e. ways of writing $n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$.

Recall that by homework 1.27, Z' is the generating function for (possibly) disconnected Gromov-Witten invariants. So $p(n)$ is really the degree n, $\chi = 0$ disconnected local Gromov-Witten invariant of E, i.e. it is the number of possible disconnected unramified covers of a torus counted with automorphisms.

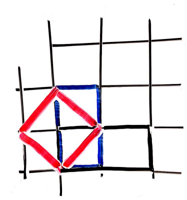


Figure 1.11: Lattice picture

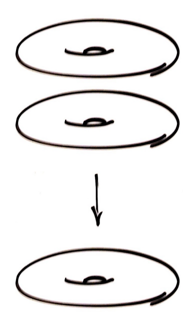


Figure 1.12: The map contributing to disconnected invariants

Example 1.34. For example with $d = 2$, we have $\sigma(2) = 3$ we can see this from figure 1.11. And $p(2)$ the cardinality of set of partitions, $\{1+1, 2\}$ so $p(2) = 2$.

Imagine ... we might suppose there are some underlying *integers* $n_{g,\beta}(X)$ (some sort of enumerative count) such that $\{N^{GW}_{g,\beta}(X)\}$ can be recovered from $n_{g,\beta}(X)$ by applying multiple-cover formulae.

Conjecture 1.35 (Gupakumar-Vafa). *There exists $n_{g,\beta}(X) \in \mathbb{Z}$ such that we may write*

$$\begin{aligned} F'(X) &= \sum_{g \geq 0, \beta \neq 0} N^{GW}_{g,\beta}(X) \lambda^{2g-2} v^{\beta} \\ &= \sum_{g \geq 0, \beta \neq 0} n_{g,\beta}(X) \sum_{k=1}^{\infty} \frac{1}{k} \left(2 \sin \left(\frac{k\lambda}{2} \right) \right)^{2g-2} v^{k\beta}. \end{aligned} \tag{1.6}$$

And moreover, $n_{g,\beta}(X) = 0$ for $g > c(\beta)$.

For example our formula implies that

$$n_{g,d}(\text{ local } \mathbb{P}^1) = \begin{cases} 1 & d = 1, g = 0 \\ 0 & \text{otherwise.} \end{cases}$$

and

$$n_{g,d}(\text{ local } E) = \begin{cases} 1 & g = 1, \text{ all } d \\ 0 & \text{otherwise.} \end{cases}$$

Gupakumar and Vafa gave a physics definition of $n_{g,\beta}(X)$. But until recently there has not been any satisfactory geometric definition. Nevertheless we can take the formula 1.6 as the *definition* of $\{n_{g,\beta}(X)\}$ in terms of $\{N^{GW}_{g,\beta}(X)\}$. The content of the conjecture is then the integrality and finiteness conjecture

$$n_{g,\beta} \in \mathbb{Z} \text{ and } n_{g,\beta} = 0 \text{ for } g > c(\beta)$$

and in fact this version of the conjecture has been proven after 15 years in 2013 using symplectic geometry: Parker and Ionel showed that using almost-complex structures this picture is not too far off. The local theory was proved by Bryan and Pandharipande. And we will talk about this later.

In some sense Gupakumar-Vafa is the most fundamental invariants among the ones we are investigation. But the issue with them is that unlike the other invariants it is hard to invent a rigorous mathematical framework for it.

Chapter 2

Donaldson-Thomas Theory

2.1 *Moduli of ideal sheaves*

Let $I_n(X, \beta)$ be the Hilbert scheme of subschemes $C \subseteq X$ with $[C] = \beta \in H_2(X, \mathbb{Z})$ and $\chi(\mathcal{O}_C) = n$. Here we use this non-conventional notation for Hilbert schemes because we really want to think about this as a space of ideal sheaves. The subscheme can be non-reduced, reducible and even can have zero dimensional components. In the case that $C \subseteq X$ is smooth and connected then of course $\chi(\mathcal{O}_X) = 1 - g$.

The same sort of anomalies that occurred in Gromov-Witten theory can also happen here as well. Recall that a smooth conic could be degenerating to a double line, and the map used to become a double cover. And now here we can also have that the image of such map is a non-reduced curve in the class β with length 2 scheme structure (figure 2.1). Here this scheme locally is $\operatorname{Spec} \mathbb{C}[x, y]/(x^2)$. The analogue of collapsing components here are embedded points on a scheme. For example in figure 2.2 the scheme locally is $\operatorname{Spec} \mathbb{C}[x, y]/(xy, x^2)$.

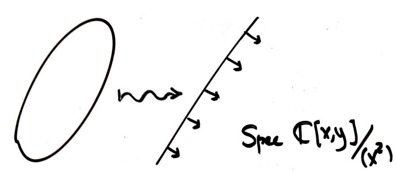


Figure 2.1: Analogue of degenerate maps

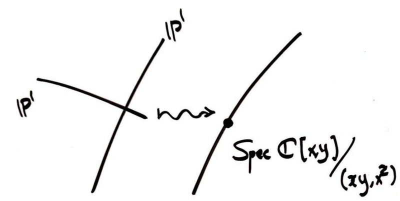


Figure 2.2: Analogue of collapsing components are embedded points

Also subschemes can be disconnected. So unlike Gromov-Witten theory, Donaldson-Thomas theory in inherently disconnected. However as we saw in homework 1.27 we can have a disconnected version of Gromov-Witten theory that turn out to have subtle connections with Donaldson-Thomas theory.

What can happen in a flat family of subschemes? But in cases of projective families there is an easy equivalent condition for the family $C_t \subseteq X$ of curves being flat; that of, $[C_t] = \beta$ and $\chi(C_t) = n$ being constant in the family.

Example 2.1. Let us look at a family of curves in $\mathbb{P}^3$. $\mathbb{P}^3$ is not a Calabi-Yau threefold but this example serves to illustrate the phenomena. So let $C_t \subseteq \mathbb{P}^3$ be a family of twisted cubics degenarating to a plane cubic. The curve $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ via $(x : y) \mapsto (x^3 : x^2y : xy^2 : y^3)$ has genus zero.

But it degenerates to a curve in the plane, with a node. But a reduced plane cubic C with a node has $\chi(\mathcal{O}_C) = 1 - g = 0$ by Riemann-Roch again. So there must have been an embedded point at the node (figure 2.3).

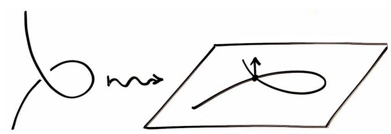


Figure 2.3: Degeneration of the cubic $(x, y) \mapsto (x^3, x^2y, xy^2, y^3)$ in $\mathbb{P}^3$ to reduced plane cubic curve C with a node

The local model is

$$C_{t\neq 0} = \{z = x = 0\} \cup \{y = z - t = 0\} \subseteq \mathbb{C}^3$$

with has ideal $(x, z)(y, z - t) = (xy, yz, xz - xt, z^2 - zt)$. The flat limit as $t \to 0$ is (xy, yz, xz, z^2). However $(xy, yz, xz, z^2) \subsetneq (z, xy)$ where the latter is the reduced structure on the locus of the degeneracy (figure 2.4).

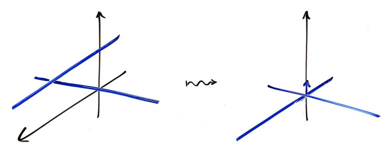


Figure 2.4: The local model of the degeneration

And this type of behavior comes as a surprise because furthermore, this family can degenerate into a nodal curve union an isolated point, and further into a smooth genus 1 curve with a point.

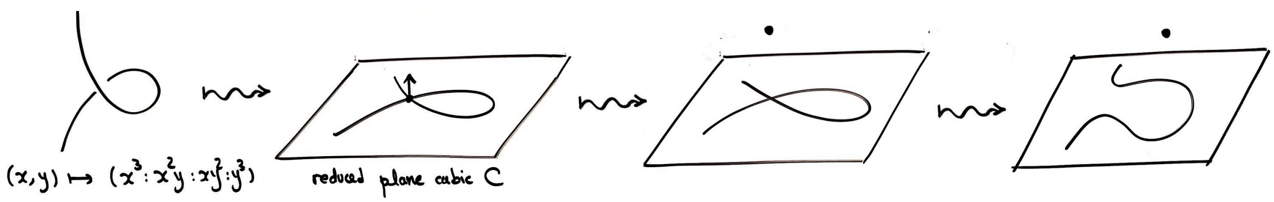


Figure 2.5: Possible deformations in families of curves. The limit can be a smooth curve with an isolated point.

For deformation theoretic reasons it is crucial that we regard $I_n(X, \beta)$ as not parametrizing subschemes but we want to regard it as parametrizing sheaves. In fact, let I be a *stable* sheaf on a Calabi-Yau threefold (and for simplicity we consider $H^1(X, \mathcal{O}_X) = 0$ in definition of a Calabi-Yau) with Chern character

$$\mathrm{ch}(I) = (1, 0, -\beta, -n) \in H^0 \oplus H^2 \oplus H^4 \oplus H^6$$

then the mapping $I \to I^{\vee\vee}$ is an embedding $I \hookrightarrow I^{\vee\vee} \cong \mathcal{O}_X$ by stability. Hence I is an ideal sheaf of some subscheme $C \subseteq X$ with $\beta = [C]$ and $\chi(\mathcal{O}_C) = n$. This tells us why we may really forget about subschemes and think of objects of our concern as sheaves. (We could have forgot about this embedding of sheaves, and consider sheaves I with fixed determinant).

For curve counting invariants we use $I_n(X, \beta)$ moduli space of ideal sheave [MNOP]. But Donaldson-Thomas theory is actually defined for any moduli space of stable sheaves. In fact, the original interest of Donaldson and Thomas were bundles.

2.2 *Digression to deformation theory of bundles*

There are different approaches to deformation theory. One is a purely algebraic approach which is too complicated for our purposed. We will consider the complex geometric approach which is very intuitive and works pretty easily for case of bundles.

Suppose $E \to X$ is a bundle given by transition functions

$$g_{\alpha\beta} : U_\alpha \cup U_\beta \to \mathrm{Gl}_n(\mathbb{C}).$$

Recall that this data has to satisfy the co-cycle condition

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \text{ on } U_\alpha \cup U_\beta \cup U_\gamma,$$

in other words, $\{g_{\alpha\beta}\} \in \check{H}(X, \mathrm{GL}_n(\mathcal{O}_X))$.

A deformation of E if given by $\{g_{\alpha\beta}(t)\}$ such that

$$g_{\alpha\beta}(t) g_{\beta\gamma}(t) g_{\gamma\alpha}(t) = 1 \text{ on } U_\alpha \cup U_\beta \cup U_\gamma, \text{ and } g_{\alpha\beta}(0) = g_{\alpha\beta}.$$

An infinitesimal deformation of E is $\{g_{\alpha\beta}(t)\}$ such that

$$g_{\alpha\beta}(t) g_{\beta\gamma}(t) g_{\gamma\alpha}(t) = 1 \mod t^2 \text{ on } U_\alpha \cup U_\beta \cup U_\gamma, \text{ and } g_{\alpha\beta}(0) = g_{\alpha\beta}.$$

Claim: Infinitesimal deformations of E are given by $h \in H^1(X, \mathrm{End}(E))$.

let $h = \{h_{\alpha\beta}\} \in \check{H}^1(X, \mathrm{End}(E))$. So

$$h_{\alpha\beta} + h_{\beta\gamma} + h_{\gamma\alpha} = 0 \text{ on } U_\alpha \cup U_\beta \cup U_\gamma.$$

But what does this condition mean? Unlike $g_{\alpha\beta}$'s, the $h_{\alpha\beta}$'s are not valued in fixed matrices, but instead it is valued in $\mathrm{End}(E|_{U_\alpha \cap U_\beta})$. To view $h_{\alpha\beta}$ is a matrix valued function, we need to choose a trivialization, by convention, we will write $h_{\alpha\beta}$ in the U_β basis. Then $h_{\alpha\beta}$ in the U_α basis is given by $g_{\alpha\beta} h_{\alpha\beta} g_{\alpha\beta}^{-1}$.

So cocycle condition written in the U_α basis reads

$$g_{\alpha\beta} h_{\alpha\beta} g_{\alpha\beta}^{-1} + g_{\alpha\gamma} h_{\beta\gamma} g_{\alpha\gamma}^{-1} + h_{\gamma\alpha} = 0.$$

Let $g_{\alpha\beta}(t) = g_{\alpha\beta}(1 + t h_{\alpha\beta} + \cdots)$ We can calculate $g_{\alpha\beta}(t) g_{\beta\gamma}(t) g_{\gamma\alpha}(t)$ up to order one, and we get

$$g_{\alpha\beta}(t) g_{\beta\gamma}(t) g_{\gamma\alpha}(t) = 1 + t(g_{\alpha\beta} h_{\alpha\beta} g_{\alpha\beta}^{-1} + g_{\alpha\gamma} h_{\beta\gamma} g_{\alpha\gamma}^{-1} + h_{\gamma\alpha} + o(t^2).$$

We can push this idea further to determine which infinitesimal deformations can be lifted to deformations of all orders. The obstructions live in $H^2(X, \mathrm{End}(E)) = \mathrm{Ob}(E)$. The moduli space of bundles near of E is given by the zero locus of a (non-linear!) map

$$H^1(X, \mathrm{End}(E)) \to H^2(X, \mathrm{End}(E)).$$

Let M be the moduli space of (stable) bundles. Then $T_{[E]} M = \mathrm{Def}(E) = H^1(X, \mathrm{End}\, E)$. Locally M is the zero locus of

$$\underbrace{\mathrm{Def}(E)}_{H^1(X,\mathrm{End}(E))} \to \underbrace{\mathrm{Ob}(E)}_{H^2(X,\mathrm{End}(E))}.$$

2.3 *Moduli spaces as critical loci of functionals*

All this picture works as well in the algebraic context of coherent sheaves, with the deformation and obstruction spaces being replaces now by

$$\begin{aligned} \mathrm{Def}(E) &\cong \mathrm{Ext}^1(E,E) \\ \mathrm{Ob}(E) &\cong \mathrm{Ext}^2(E,E) \end{aligned}$$

Using Serre duality

$$\mathrm{Ext}^i(F,G) \cong \mathrm{Ext}^{\dim X - i}(G, F \otimes K_X)^\vee.$$

in case that X is moreover a Calabi-Yau threefold we have $\mathrm{Ext}^1(E,E) \cong \mathrm{Ext}^2(E,E)^\vee$, i.e. the spaces of deformations and obstructions are dual.

Feb 3

Last time we found that for $E \to X$ a holomorphic bundle on a complex manifold, infinitesimal deformations of E are classified by $H^1(X, \mathrm{End}(E))$:

$$T_{[E]}M = H^1(X, \mathrm{End}(E)),$$

where M is the moduli space of bundles.

Obstructions to lifting infinitesimal deformations to infinite order lie in $H^2(X, \mathrm{End}(E))$. Locally the moduli space is the zero locus of a function

$$\kappa : H^1(X, \mathrm{End}(E)) \to H^2(X, \mathrm{End}(E))$$

known as the *Kuranishi map*.

Specific of Calabi-Yau threefolds is that by Serre duality,

$$H^1(X, E \otimes E^*) \cong H^2(X, (E \otimes E^*)^* \otimes K_X)^\vee$$

so the space of deformations is

$$\mathrm{Def} = H^2(X, \mathrm{End}(E))^\vee = (\mathrm{Ob})^\vee.$$

So we may view κ as a sectino of $T^*H^1(X, \mathrm{End}(E)) \to H^1(X, \mathrm{End}(E))$. In other words, κ is a 1-form which is necessarily exact, $\kappa = df$. Hence f is characterizing the moduli space M locally near $[E]$ as the critical locus of f,

$$M \cong \mathrm{crit}(f).$$

This map $f : H^1(X, \mathrm{End}(E)) \to \mathbb{C}$ is know as the *Chern-Simon (super-)potential*.

A priori f is the germ of a holomorphic function, or in the algebraic context, a formal function. But recently is has been proven in cases that it is realized as a polynomial.

More generally, if E is now a coherent sheaf and $\mathcal{M}$ is the moduli space of sheaves, then $T_{[E]}\mathcal{M} = \mathrm{Ext}^1(E,E)$ and $\mathrm{Ob}(E) = \mathrm{Ext}^2(E,E)$. And the Kuranishi map now has an algebraic version

$$\kappa : \mathrm{Ext}^1(E,E) \to \mathrm{Ext}^2(E,E) \cong \mathrm{Ext}^1(E,E)^\vee.$$

So locally

$$\mathcal{M} \cong \mathrm{crit}(f : \mathrm{Ext}^1(E,E) \to \mathbb{C}).$$

2.4 *Donaldons-Thomas invariants*

If $\mathcal{M}$ is a compact moduli space of sheaves on a Calabi-Yau threefolds, as in example of $I_n(X,\beta)$, (this usually involves a choice of a stability condition then $[\mathcal{M}]^{vir} \in H_0(\mathcal{M},\mathbb{Z})$ as hence we obtain a numerical invariant

$$N_M^{DT}(X) = \int_{[\mathcal{M}]^{vir}} 1 \in \mathbb{Z}.$$

In particular our curve counting invariants are

$$N_{n,\beta}^{DT}(X) = \int_{[I_n(X,\beta)]^{vir}} 1$$

which should be thought of a virtual count of degree β curves C in X with $\chi(\mathcal{O}_C) = n$.

Suppose M is a compact moduli space of sheaves on a Calabi-Yau threefold, and suppose $\mathcal{M}$ is smooth of dimension d. Then the obstruction sheaf $Ob \to \mathcal{M}$ is a bundle of rank d, then $\mathrm{Ob} \cong T^*M$. So the virtual class is the top Chern class of the cotangent bundle:

$$\int_{[M]^{vir}} 1 = \int_{[M]} c_d(T^*M) = (-1)^d \int_{[M]} d_d(TM) = (-1)^d e(M),$$

where $e(M)$ is the top Euler character of M.

Now the amazing fact is that this behavior as an Euler characteristic, is true even in vase $\mathcal{M}$ is not a smooth space:

Theorem 2.2 (Behrend)**.**

$$\int_{[M]^{vir}} 1 = \sum_{k\in\mathbb{Z}} ke(\nu_M^{-1}(k)) =: e_{vir}(M) \tag{2.1}$$

where $\nu_M : M \to \mathbb{Z}$ *is the* behrend function *defined by*

$$\nu_M([E]) = (-1)^{\dim \mathrm{Ext}^1(E,E)}(1 - e(MF_f))$$

where MF_f *is the Milnor fiber of the Chern-Simons superpotential,* f*.*

If $[E] \in \mathcal{M}$ is a smooth point of $\mathcal{M}$ then

$$\nu_M([E]) = (-1)^{\dim \mathrm{Ext}^1(E,E)}.$$

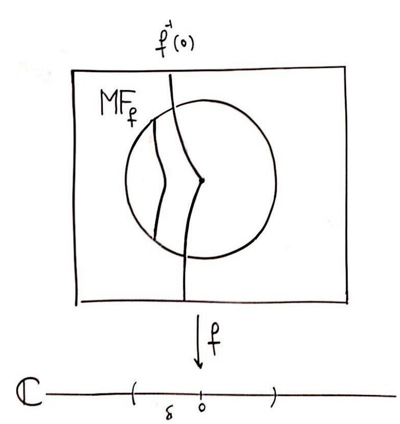


Figure 2.6: Milnor fiber

Milnor fiber of $f : \mathrm{Ext}^1(E,E) \to \mathbb{C}$ is by definition

$$MF_f = \{f^{-1}(\delta) \cap B_\varepsilon : 0 < \delta \ll \varepsilon \ll 1\}.$$

As quick examples if M is smooth and $f = 0$ then $MF_f = \emptyset$. If $M = \mathrm{Spec}\{\mathbb{C}[x]/x^n\}$, if the fat point of length n, in which case $f : \mathbb{C} \to \mathbb{C}$ is $x \mapsto x^{n+1}$, then $M = \{df = 0\} = \{x^n = 0\}$.[1]

The Milnor fiber in this example is

$$MF_f = \{x^{n+1} = \delta\} \cap B\varepsilon \cong \{x^{n+1} \cong 1\}.$$

[1] For simple spaces, structure of M determines the potential f. But in general one needs to investigate the Kuranishi map to understand f. In fact f is not uniquely determined by M in general however one advantage of Behrend's function is that it is uniquely determined by M.

Let us compact the different paradigms the two sides of formula 2.1. The left hand side is an integral over a moduli space, whereas the right hand side is the euler characteristic of some sort. In fact

1. The right hand side is define even if M is non-compact (e.g. If X is a non-compact Calabi-Yau threefold). So we may use this right hand side, more generally as our *definitions* for invariants in the more general context.
2. Euler characteristic can be computer motivicaly:

   $$e(X) = e(X \setminus Z) + e(Z) \quad \text{if } Z \subseteq X \text{ is a closed set, and also}$$
   $$e(X_1 \times X_2) = e(X_1)e(X_2).$$

3. By localization if $\mathcal{M}$ admits a $\mathbb{C}^*$-action then

   $$e(M) = e(M^{\mathbb{C}^*}),$$

   and this follows from the previous motivic property of these invariants.
4. Categorification: the ordinary euler characteristics are alternating sums of Betti numbers: $e(M) = \sum(-1)^n \dim H^n(M)$. This suggests existence of some cohomology theory $\widetilde{H}^*(M)$ such that $e_{vir}(M) = \sum(-1)^n \dim \widetilde{H}^n(M)$. Existance of $\widetilde{H}^*(M)$, implies a categorification of Donaldson-Thomas invariants associated to M; that is a transition from the set of numbers (where the numerical invariants live) to the category of vector spaces (where the cohomological invariants live).

However, there is essentially one important feature that can be seen on the left hand side the formula 2.1 and not the right hand side, and that is deformation invariance!

Feb 5

Last time we saw that that we can use weighted topological Euler characteristics instead of virtual classes in Donaldson-Thomas theory. We talked about $I_n(X,\beta)$, the Hilbert scheme of subschemes $C \subseteq X$ where $[C] = \beta$, $n = \chi(\mathcal{O}_X)$ and viewed is as moduli space of ideal sheaves:

$$N^{DT}_{n,\beta}(X) = \int_{[I_n(X,\beta)]^{vir}} 1 = e_{vir}(I_n(X,\beta)).$$

Example 2.3. We consider the local $\mathbb{P}^2$: $X = \mathrm{tot}(\mathcal{O}(-3) \to \mathbb{P}^2)$. Let $\ell = [\mathbb{P}^1]$ be the class of a line. Then the lines on the base (figure 2.7) are the only curves that contribute to:

$$I_1(X, \ell) = \{ \text{ lines in } \mathbb{P}^2 \} \cong \mathbb{P}^2.$$

So $N^{DT}_{1,\ell}(X) = e(\mathbb{P}^2) = 3$. Similarly, in degree two the conics (figure 2.8)

$$I_1(X, 2\ell) = \{ \text{ conics in } \mathbb{P}^2 \} = \mathbb{P}^5.$$

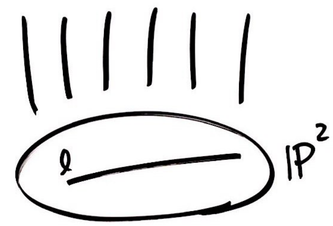


Figure 2.7: Lines in $\mathbb{P}^2$

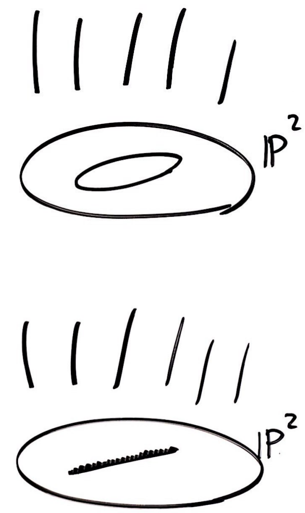
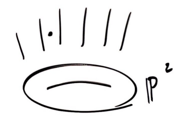


Figure 2.8: Conics in $\mathbb{P}^2$

So $N^{DT}_{1,2\ell}(X) = (-1)e(\mathbb{P}^5) = -6$. In higher euler characteristics things get more complicated:

$I_2(X, \ell) = \Bigg\{$  $\Bigg\}$

This space is more complicated. Firstly, unlike the previous items this space is non-compact. Although it is easy to see that it is 5 dimensional (the isolated point lives in the three dimensional total space and the line lives in the zero section). However, this space turns out to be smooth and hence

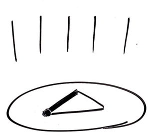
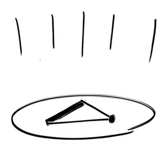
$$N^{DT}_{2,\ell}(X) = (-1)e(I_n(X, \ell)).$$

One technique for computing on this space is using a torus action. Let $T = (\mathbb{C}^\times)^3$ act on $I_n(X, \ell)$ in this manner: we lift an action of a torus T on $\mathbb{P}^2$ to an action on $\mathcal{O}(-3) \to \mathbb{P}^2$, i.e. on X, such that the action is nontrivial on the fibers of the fixed loci of $\mathbb{P}^2$. This induces an action of T on $I_n(X, d\ell)$.

Now for any space Y, with a torus action, $e(Y) = e(Y^T)$. Therefore

$$I_n(X, d\ell)^T = \{T \text{ invariant subschemes } C \subseteq X\}.$$

So

$I_2(X, \ell)^T = \Bigg\{$ 6 types like  $\Bigg\} \cup \Bigg\{$ 3 points like  $\Bigg\}$.

And hence

$$N^{DT}_{2,\ell}(X) = (-1)(6)\#\left\{ Z \subseteq \mathbb{C}^3, \begin{array}{c} \text{T-invariant, supported on } z\text{-axis} \\ \text{with embedded point at the origin} \end{array} \right\} - 3$$

$$= -3 - 6.\#\left\{ I \subseteq \mathbb{C}[x, y, z], \begin{array}{c} I \text{ is generated by monomials} \\ \sqrt{I} = (x, y), \dim \sqrt{I}/I = 1 \end{array} \right\}.$$

To do this count we visualize monomials as boxed in the positive octant of the 3-space. Appearance of a monomial $x^i y^j z^k$ in generators, accounts to removing a shifted octant form the a configuration of boxes. Our favorite configurations are the ones that asymptotically only have boxes along the z-axis. Fat points at the origin correspond to a cummulate of finitely many boxes in neighbourhood of the origin. Here we only have two possibilities as shown in figure 2.9.

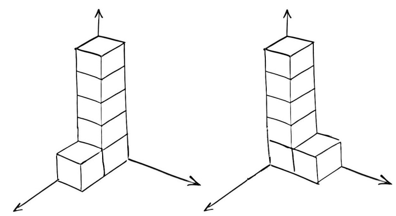


Figure 2.9: Configurations that asymptotically have one box along the z-axis

respectively corresponding to (xz, x^2, y) and (x, y^2, zy). So we conclude that $N^{DT}_{2,\ell}(X) = -3 - 6.2 = -15$. In the presence of a T-action we can compute N^{DT} by box counting. We will do this systematically when we study topological vertex.

Remark 2.4. In Gromov-Witten theory, we also have a localization technique. There we are not dealing with euler characteristics and the situation is more complicated, namely that of Atiyah-Bott localization. For example,

$$\overline{\mathcal{M}}_1(X, \ell)^T = \text{ 6 copies of } \overline{\mathcal{M}}_{1,1}$$

as hinted by in the figure 2.10). So $\overline{\mathcal{M}}_g(X, d\ell)^T$ can be written in terms of $\overline{\mathcal{M}}_{g,d}$.

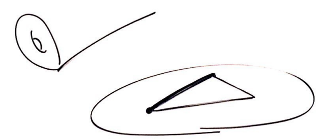


Figure 2.10: Atiyah-Bott localization

A few more examples:

Example 2.5. Let X be the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$ this time.

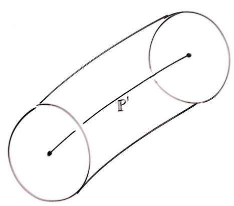


Figure 2.11: The local $\mathbb{P}^1$

This space has a T-action with 2 fixed points on a fixed curve. Obviously $N^{DT}_{1,\ell}(X) = 1$. We have $N^{DT}_{2,\ell}(X) = (-1).\#I_2(X, \ell)^T = -4$ counted by configurations of the type shown in figure 2.12.

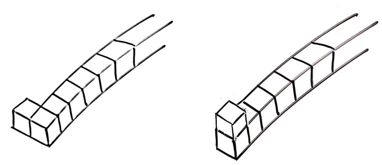


Figure 2.12: Configurations contributing to $I_2(X, \ell)$

And $N^{DT}_{3,\ell}(X) = \#I_3(X, \ell)^T = 14$ is counted by following types of configurations (figure 2.13).

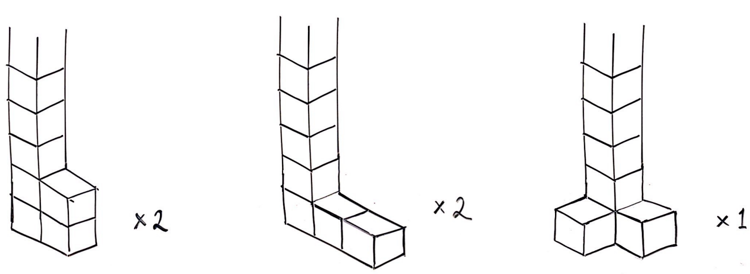


Figure 2.13: Configurations contributing to $I_3(X, \ell)$

Feb 7

2.5 GW/DT correspondence

Recall that $N^{DT}_{n,\beta}(X) = \int_{[I_n(X,\beta)]^{vir}} 1 = e_{vir}(I_n(X,\beta))$. The partition function for Donaldson-Thomas invariants is defined as

$$Z^{DT}_\beta(X) = \sum_n N^{DT}_{n,\beta}(X)q^n \in \mathbb{Z}[q^{-1}][\![q]\!].$$

Recall that

$$\begin{aligned}
F^{GW}_g(X) &= \sum_\beta N^{GW}_{g,\beta}(X)v^\beta \\
(F^{GW}_g(X))' &= \sum_{\beta\neq 0} N^{GW}_{g,\beta}(X)v^\beta \\
F^{GW}(X) &= \sum_g F^{GW}_g(X)\lambda^{2g-2} \\
(F^{GW}(X))' &= \sum_g (F^{GW}_g(X))'\lambda^{2g-2} \\
Z^{GW}(X) &= \exp(F^{GW}(X)) \\
(Z^{GW}(X))' &= \exp(F') = \exp(F - F|_{v=0})
\end{aligned}$$

From homework we have seen an interpretation of the coefficients of $Z^{GW}(X)$ and $(Z^{GW}(X))'$. In fact

$$Z^{GW}(X)' = \sum_{\beta,\chi} N^{\bullet,GW}_{\beta,\chi}(X)' v^\beta \lambda^{-2\chi}.$$

hence $N^\bullet_{\beta,X}$ has a geometric interpretation too: it is the virtual count of possibly disconnected stable maps of degree β and $\chi(C) = \chi$ such that no connected component collapses.

This suggests that we should be trying to relate $Z^{GW}(X)'$ and $Z^{DT}(X)'$. Here we define $Z^{DT}(X)'$ as

$$Z^{DT}(X)' = \frac{Z^{DT}(X)}{Z^{DT}_0(X)}.$$

This fraction has no obvious geometric interpretation. Loosely speaking we do not allow points away from curves.

Conjecture 2.6 (MNOP)**.**

$$Z^{DT}_\beta(X)' = Z^{GW}_\beta(X)'$$

after the change of variables $q = -e^{i\lambda}$.

This is a priori a very bizarre change of variables! In fact for the conjecture to even make sense we need

Conjecture 2.7. *$Z^{DT}_\beta(X)'$, a priori in $\mathbb{Z}[q^{-1}][\![q]\!]$ is a rational function in q invariant under $q \leftrightarrow q^{-1}$, i.e. it is linear combination of $\frac{q}{(1-q)^2}, \frac{q^k}{(1-q^k)^2}, \dots$.*

Because this change of variables is happening through analytic continuation, in order to prove this conjecture, we cannot compare invariants number by number but we need to get a single Gromov-Witten invariant $N^{GW}_{\beta,g}$ we must know $N^{DT}_{\beta,n}$ for all n. To get a single Gromov-Witten invariant $N^{GW}_{\beta,g}$ we must know $N^{DT}_{\beta,n}$ for all n.

Remark 2.8. $Z^{GW}_\beta(X)'$ is an expansion about $\lambda = 0$ and $Z^{DT}_\beta(X)'$ is an expansion about $q = 0$, i.e. as $\lambda \to i\infty$. In physics jargon, this is called non-perturbative duality. $Z^{GW}_\beta(X)$ is the perturbative expansion at weal string coupling and $Z^{DT}_\beta(X)'$ is the perturbative expansion at strong string couplings. These partition functions are energies of the vacuum in different theories. The Gromov-Witten partition function is related to Type A topological string theory and the Donaldson-Thomas one is related to Type B topological string theory (with D0-D2-D6 branes). The relation between the two is known as S-duality.

Example 2.9. Let us observe this rationality conjecture in case of the resolved conifold, a.k.a. the local $\mathbb{P}^1$:

$$X = \mathrm{tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1).$$

Recall that

$$F^{GW}(X)' = \sum_{d>0} \frac{v^d}{d}\left(2\sin\left(\frac{d\lambda}{2}\right)\right)^{-2}.$$

So setting $Q = e^{i\lambda}$ we get that

$$\begin{aligned} F^{GW}(X)' &= \sum_{d>0} -\frac{v^d}{d}((e^{-id\lambda/2})(e^{id\lambda}-1))^{-2} \\ &= \sum_{d>0} -\frac{v^d}{d}\frac{Q^d}{(1-Q^d)^2} \\ &= \sum_{d>0} -\frac{v^d}{d}\sum_{m>0} mQ^{md} \\ &= \sum_{m=1}^{\infty} m \sum_{d=1}^{\infty} -\frac{(vQ^m)^d}{d} \\ &= \sum_{m=1}^{\infty} m\log(1-vQ^m) \end{aligned}$$

So

$$F^{GW}(X)' = \log\left(\prod_{m=1}^{\infty}(1-vQ^m)^m\right).$$

And recall that

$$Z^{GW}(X)' = \prod_{m=1}^{\infty}(1-v(-q)^m)^m = Z^{DT}(X)'$$

under our change of variables.

Part of the goal today is to examine this conjecture in a few cases.
Example 2.10. The $\beta = [\mathbb{P}^1]$-term:

$$Z'_{[\mathbb{P}^1]}(X) = \text{ coefficient of } v^1 \text{ in } \prod_{m=1}^{\infty}(1 - v(-q)^m)^m$$

But we have

$$\begin{aligned}\prod_{m=1}^{\infty}(1 - v(-q)^m)^m &= \prod_{m=1}^{\infty}(1 - mv(-q)^m + O(v^2)) \\ &= 1 - \sum_{m=1}^{\infty} mv(-q)^m + O(v^2)\end{aligned}$$

So

$$Z'_{[\mathbb{P}^1]}(X) = \sum_{m=1}^{\infty} -m(-q)^m = q - 2q^2 + 3q^3 + \cdots .$$

So the degree-one prediction here is that

$$q - 2q^2 + 3q^3 - \cdots = \frac{\sum_{n=1}^{\infty} N^{DT}_{n,[\mathbb{P}^1]}(X)q^n}{\sum_{n=0}^{\infty} N^{DT}_{n,0}(X)q^n}.$$

But we already know that

$$\begin{aligned}N^{DT}_{n,0}(X) &= e_{vir}(\mathrm{Hilb}_n(X)) \\ N^{DT}_{0,0}(X) &= 1 \quad \text{the ``ideal'' sheaf in } \mathcal{O}_X \\ N^{DT}_{1,0}(X) &= e_{vir}(X) = -e(X) = -2. \\ N^{DT}_{1,[\mathbb{P}^1]}(X) &= e_{vir}(I_1(X,[\mathbb{P}^1])) = 1 \\ N^{DT}_{2,[\mathbb{P}^1]}(X) &= e_{vir}(I_2(X,[\mathbb{P}^1])) = (-1) \qquad \underbrace{e(I_2(X,[\mathbb{P}^1])^T)}_{\text{4 fixed points of torus action}}\end{aligned}$$

So the conjecture predicts that

$$(q - 2q^2 + 3q^3 - \cdots)(1 - 2q + \cdots) = (q - 4q^2 + \cdots)$$

It is already miraculous that the first factor on left hand side has integer coefficients. The fact that the two series start both at the same term is already also miraculous from point of view of Gromov-Witten theory.

To use the torus action to compute $e_{vir}(I_n(X,\beta))$ we need to know the value of Behrend function at the fixed points of the action.

Feb 12

Suppose X is a toric Calabi-Yau threefold. Such a variety is a $T = (\mathbb{C}^\times)^3$ equivariant partial compactification of $(\mathbb{C}^\times)^3$. So T acts on X with isolated fixed points. And there are T-equivariant coordinates about each fixed point.

The induced action of T on $I_n(X,\beta)$ has isolated fixed points corresponding to subschemes $Z \subset X$ which are T-invariant. These are defined by monomial ideals on each coordinate patch. And are also in one-to-one correspondence with "piles of boxes" in the positive octant. There is a 2-dimensional subtorus $T' \subset T$ which acts trivially on $K_X = X \times \mathbb{C}$. It is a little harder to see that

$$I_n(X,\beta)^T = I_n(X,\beta)^{T'}.$$

For fixed points of such group actions, the group acts on the tangent space. So if $[I] \in I_n(X,\beta)^{T'}$, then T' acts on $\mathrm{Def}(I) = \mathrm{Ext}^1(I,I)$ and on $\mathrm{Ob}(I) = \mathrm{Ext}^2(I,I)$ and the Kuranishi map κ is T'-equivariant:

$$\kappa : \mathrm{Ext}^1(I,I) \to \mathrm{Ext}^2(I,I).$$

Remember that $\kappa = df$ for f the Chen-Simons potential $f : \mathrm{Ext}^1(I,I) \to \mathbb{C}$. This implies that f is T'-invariant. The claim is that this implies that $e(MF_f) = 0$.

Recall that $\nu([I]) = (-1)^{\dim \mathrm{Ext}^1(I,I)}(1 - e(MF_f))$. Here

$$MF_f = \{f^{-1}(\delta) \cap B_\varepsilon : 0 < \delta \ll \varepsilon \ll 1\}.$$

We may fix a Hermitian metric on $\mathrm{Ext}^1(I,I)$ (we need to do that anyways to have the balls defined by that metric but) in such a way that $S^1 \times S^1 \subset T^1 \cong \mathbb{C}^\times \times \mathbb{C}^\times$ acts unitarily. Then $S^1 \times S^1$ acts freely on MF_f. This implies $e(MF_f) = 0$. Consequently $\nu([I]) = (-1)^{\dim \mathrm{Ext}^1(I,I)}$.

Proposition 2.11 (MNOP)**.** *Let $I \in I_n(X,\beta)^T$, then $\dim \mathrm{Ext}^1(I,I) \mod 2$, depends on β and linearly on $n \mod 2$.*

This implies that

$$Z^{DT}_\beta(X) = \sum_n e_{vir}(I_n(X,\beta))q^n = \pm \sum_n (-1)^n \# I_n(X,\beta)^T q^n.$$

Example 2.12. Let $x_1, \cdots, x_F$ be the fixed points of X. And recall that locally $\mathrm{Hilb}^n_{x_i}(X)$ is the same moduli space as $\mathrm{Hilb}^n_0(\mathbb{C}^3)$. So,

$$\begin{aligned} Z^{DT}_0(X) &= \sum_{n=0}^{\infty} (-1)^n \# \mathrm{Hilb}^n(X)^T q^n \\ &= \sum_n \sum_{n_1+\cdots+n_F=n} \prod_{i=1}^F \left(\mathrm{Hilb}^{n_i}_{x_i}(X)^T(-q)^{n_i}\right) \\ &= \left(\sum_n \# \mathrm{Hilb}^n_0(\mathbb{C}^3)(-q)^n\right)^F = M(-q)^F. \end{aligned}$$

Here $M(q) = \sum_{n=0}^\infty (\#\text{ 3d-partitions of } n)q^n = \prod_{m=1}^\infty (1-q^m)^{-m}$. Notice also that F is the topological euler characteristic of X, hence we conclude that $Z_0(X) = M(-q)^{e(X)}$.

Theorem 2.13. $Z_0(X) = M(-q)^{e(X)}$ *even if the Calabi-Yau threefold X is not toric.*

Example 2.14. $X = \mathrm{tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1)$. The prediction from Gromov-Witten theory was that

$$\frac{Z^{DT}(X)}{Z_0^{DT}(X)} = \prod_{m=1}^{\infty} (1 - v(-q)^m)^m.$$

The v-term is

$$\frac{Z_1^{DT}(X)}{Z_0^{DT}(X)} = \frac{q}{(1+q)^2} = q - 2q^2 + 3q^3 - 4q^4 + \cdots .$$

The MacMahon function function is $M(q) = q + 1 + 3q^2 + 6q^3 + 14q^4 + \cdots$ And one can check that the identity

$$\frac{q - 4q^2 + 14q^3 - 42q^4 + \cdots}{(1 - q + 3q^2 - 6q^3 + \cdots)^2} = q - 2q^2 + 3q^3 + 4q^4 + \cdots$$

holds in the first few terms.

Feb 14

2.6 *Topological vertex*

For same space we now want to compute Donaldson-Thomas invariants by counting $I_n(X, d[\mathbb{P}^1])^T$, i.e. counting subschemes by monomial ideals in each coordinate patch, and i.e. box configurations of the following form. $[Z] \in I_n(X, d[\mathbb{P}^1])^T$, such that $Z \cap \mathbb{C}^2 \subset \mathbb{C}^2$ of length d is invariant under $(\mathbb{C}^\times)^2$-action on $\mathbb{C}^2$ where $\mathbb{C}^2$ is considered is the generic fiber of $X \to \mathbb{P}^1$. In other words it is given by a 2D partition λ of d.

Recall that $\lambda \vdash d$ if $\lambda = (\lambda_1, \cdots, \lambda_\ell)$ is a sequence of positive integers satisfies $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ and $|\lambda| = \sum_{i=1}^{\ell} \lambda_i = d$. We picture this as d boxes in the positive quadrant as in figure 2.14 is such a way that $\mathbb{C}[x,y]/I_\lambda$ has a basis as a $\mathbb{C}$-vector space labelled by every $\square \in \lambda$. So for instance in the above picture we have $\lambda = (5,4,1,1)$ and $I_\lambda = (y^5, xy^4, x^2y, x^4)$.

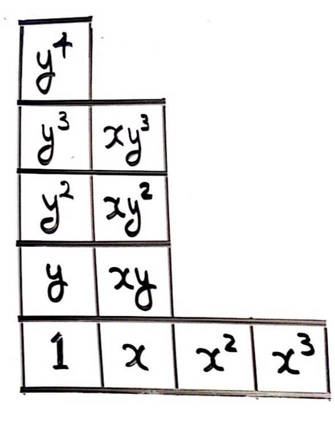


Figure 2.14: Interpretation of a 2D-partition as a quotient ring.

Let $Z_\lambda \subset X$ be the T-fixed subscheme with cross-section given by λ and no embedded points. In general $I_n(X, d[\mathbb{P}^1])^T$ parametrizes subschemes that look like Z_λ for some $\lambda \vdash d$ with embedded points at 0 and ∞. In each coordinate chart our subscheme is determined by a 3D partition in the positive octant, asymptotic to λ in the z-direction (figure 2.15).

There exists a combinatorial tool that counts these configurations of boxes and it is called the topological vertex:

Definition 2.15. $V_{\emptyset\emptyset\lambda}(q) = \sum_\pi q^{|\pi|}$ where the summation ranges overs all 3D partitions that are asymptotic to λ and $|\pi|$ is the number of added boxes (i.e. the orange boxes in pictures like above).

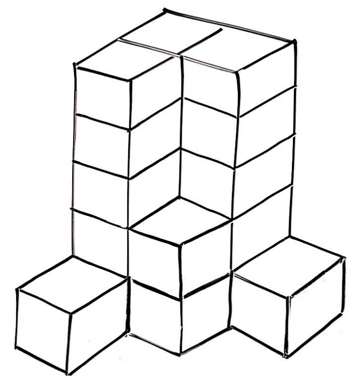


Figure 2.15: The asymptote of λ has contains three cubes in z-direction.

Theorem 2.16 (Okounkov, Reshetikhin, Vafa)**.**

$$V_{\emptyset\emptyset\lambda}(q) = M(q)q^{-\binom{\lambda}{2}} S_{\lambda^t}(1, q, q^2, q^3, \cdots).$$

Here $M(q)$ is the usual MacMahon function

$$M(q) = \prod_{m=1}^{\infty}(1-q^m)^{-m} = V_{\emptyset\emptyset\emptyset}(q)$$

and the notation $\binom{\lambda}{2}$ is really the summation $\sum_i \binom{\lambda_i}{2}$. λ^t is the transpose of λ defined in the obvious way.

And $S_{\lambda^t}(x_1, x_2, \cdots)$ is the Schur function labelled by λ. It is a symmetric function homogeneous of degree $|\lambda|$. And S_λ for a basis for symmetric function. In particular if $\lambda \vdash 1$ then $\lambda = \Box$ and

$$S_\Box(x_1, x_2, \cdots) = x_1 + x_2 + \cdots$$

hence

$$S_\Box(1, q, q^2, \cdots) = 1 + q + q^2 + \cdots = \frac{1}{1-q}.$$

In fact it turns out that Schur functions are all rational. One useful identity that holds for Schur functions is

$$\sum_\lambda S_\lambda(x_1, x_2, \cdots) S_{\lambda^t}(y_1, y_2, \cdots) = \prod_{i,j}(1 + x_i y_j).$$

Since $V_{\emptyset\emptyset\lambda} = V_{\emptyset\emptyset\lambda^t}$ then

$$q^{-\binom{\lambda}{2}} S_{\lambda^t}(1, q, q^2, \cdots) = q^{-\binom{\lambda^t}{2}} S_\lambda(1, q, q^2, \cdots).$$

In fact this is equal to $\prod_{\Box\in\lambda} \frac{1}{1-q^{h(\Box)}}$ where $h(\Box)$ is the number of boxes in a hook.

Theorem 2.17 (MNOP)**.** *For any* $[I_Z] \in I_n(X, d[\mathbb{P}^1])$ *we have*

$$\dim \operatorname{Ext}^1(I_Z, I_Z) = d + n \mod 2$$

and consequently, $\nu([I_Z]) = (-1)^{d+n}$.

Calculation in same reference shows that

$$\chi(\mathcal{O}_{Z_\lambda}) = |\lambda| + \binom{\lambda}{2} + \binom{\lambda^t}{2}.$$

So we can now compute the DT partition function

$$\begin{aligned} Z^{DT}(X) &= \sum_{d=0}^{\infty}\sum_n N^{DT}_{n,d[\mathbb{P}^1]}(X) v^d q^n \\ &= \sum_{d=0}^{\infty}(-1)^d v^d \sum_{\lambda\vdash d} V_{\emptyset\emptyset\lambda}(-q) V_{\emptyset\emptyset\lambda}(-q)(-q)^{|\lambda|+\binom{\lambda}{2}+\binom{\lambda^t}{2}} \end{aligned}$$

By a change of variables we have

$$
\begin{aligned}
Z^{DT}(X)(-q,-v) &= \sum_\lambda v^{|\lambda|} V_{\emptyset\emptyset\lambda}(q)^2 (q)^{|\lambda| + \binom{\lambda}{2} + \binom{\lambda^t}{2}} q^{-2\binom{\lambda}{2}} \\
&= \underbrace{M(q)^2}_{Z_0^{DT}(X)(-q)} \sum_\lambda v^{|\lambda|} S_\lambda(q) S_\lambda(q) q^{|\lambda|} q^{\binom{\lambda}{2}} q^{\binom{\lambda^t}{2}}
\end{aligned}
$$

And remember we were really interested in the reduced Donaldson-Thomas theory:

$$
\begin{aligned}
Z^{DT}(X)(-q,-v)' = \frac{Z^{DT}(X)(-q,-v)}{Z_0^{DT}(X)(-q)} &= \sum_\lambda (qv)^{|\lambda|} S_\lambda(q) S_{\lambda^t}(q) \\
&= \sum_\lambda S_\lambda(qv, q^2 v, q^3 v, \cdots) S_{\lambda^t}(1, q, q^2, \cdots) \\
&= \prod_{i,j \geq 1} (1 + vq^{i+j-1}) \\
&= \prod_{m=1}^{\infty} (1 + vq^m)^m .
\end{aligned}
$$

And we conclude that

$$
Z^{DT}(X)' = \prod_{m=1}^{\infty} (1 - v(-q)^m)^m
$$

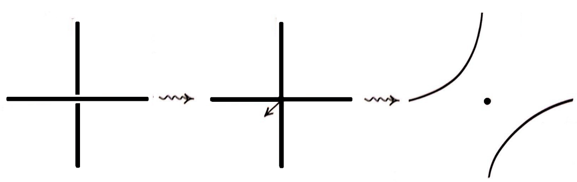
as predicted.

Chapter 3

Pandharipande-Thomas Theory

Feb 26

Let's recall some of the basic problematic phenomena in Donaldson-Thomas theory. We saw that smooth genus g curves can deform in families to nodal curves with embedded points and further to smooth genus $g+1$ curves together with a point:



The Hilbert scheme aka the moduli space of ideal sheaves includes floating points! This is undesirable for curve counting. In fact because of this, the MNOP conjecture needed to remove point contributions formally via

$$Z^{DT}(X)' = \frac{Z^{Dt}(X)}{Z^{DT}(X)|_{v=0}} = M(-q)^{-e(X)} Z^{DT}(X).$$

We want a theory that revives the right hand side $M(-q)^{-e(X)}Z^{DT}(X)$ somehow "on its own".

So the question of better compactification is: are there limits that do not create embedded points? For instance, let $C_t = C_t^0 \cup C_t^1$ for all $t \neq 0$ as in the following figure 3.1

Note that in level of ideal sheaves: the flat limit of I_{C_t} as $t \to 0$ is not the same as $I_{C_0^1 \cup C_1^0}$. What about $\lim_{t\to 0} \mathscr{O}_{C_t}$? This one is the same as

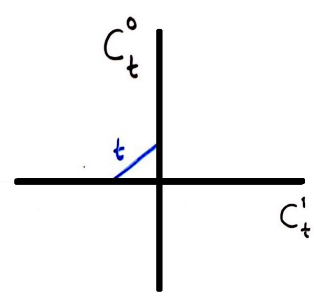


Figure 3.1: Are there limits that do not create embedded points?

$\lim_{t\to 0} \mathcal{O}_{C_t^0} \oplus \mathcal{O}_{C_t^1}$, in other words the sheave $\mathcal{O}_{C_0^1} \oplus \mathcal{O}_{C_0^0}$ is a flat limit of $\mathcal{O}_{C_t}$. One can conclude that maybe we need to work with these types of sheaves rather than ideal sheaves.

The first issue with this, is that flat limits are not unique. There are non-trivial extensions

$$0 \to \mathcal{O}_{C_0^1} \to F \to \mathcal{O}_{C_0^0} \to 0$$

which occur as flat limits. This tells us that the moduli space is not separated.

The second issue is that the extensions of $\mathcal{O}_{C_0^0}$ by $\mathcal{O}_{C_0^1}$ have more complicated automorphism groups. Sheaves like $\mathcal{O}_{C_t^0} \oplus \mathcal{O}_{C_t^1}$ have automorphism groups which can vary in moduli. The upshot of this is that the moduli space of such sheaves is a (non-separated) Artin stack.

We need to "remember" that $\mathcal{O}_{C_t}$ is a structure sheave; we have a pair consisting of a sheaf and a surjective map f from $\mathcal{O}_X$:

$$f : \mathcal{O}_X \to \mathcal{O}_{C_t}.$$

But f can have limits that are not surjective! And the definition of stable pairs due to Pandharipande and Thomas is the right way to dream a moduli space for these objects that is proper (the limits exists) and is separated (the limits are unique).

Definition 3.1 (Pandharipande-Thomas)**.** Let X be a threefold. A stable pair $(\mathcal{F}, s)$ is a sheaf $\mathcal{F}$ of dimension 1 and a map $\mathcal{O}_X \xrightarrow{s} \mathcal{F}$ (hence a section $s \in H^0(X, \mathcal{F})$) such that

1. $\mathcal{F}$ is pure (i.e. there does not exist any non-zero $\mathcal{G} \to \mathcal{F}$ such that $\dim \mathcal{G} = 0$); and,
2. The $\mathrm{coker}(s)$ is dimension zero (so s cannot vanish on a curve).

The first condition eliminates possibility of existence of embedded points in case of curves for example (this is equivalent to the sheaf being Cohen-Macauly in case of curves). The second condition above is as much as we want s to be close to a surjection. The natural notion of equivalence of pairs is that, two pairs $s : \mathcal{O}_X \to \mathcal{F}$ and $s : \mathcal{O}_X \to \mathcal{F}'$ are equivalent if there exists an isomorphism $\mathcal{F} \cong \mathcal{F}'$ making the following triangle commute:

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "s"] \arrow[dr, "s'"'] & \mathcal{F} \arrow[d, "\cong"] \\ & \mathcal{F}' \end{tikzcd}$$

In this case our discrete invariants are $n = \chi(\mathcal{F})$ and note that $\ker(s) = I_C$, for $[C] = \beta \in H_2(X)$. The above pairs had been previously studied, and the main theorem that tells us this is the correct definition is the following

Theorem 3.2 (LePotier, Pandharipande-Thomas)**.** *The moduli space of stable pairs with fixed n, β is a proper and separated schemes:*

$$PT_n(X, \beta) = \{ \textit{ stable pairs } \mathcal{O}_X \xrightarrow{s} \mathcal{F}; n = \chi(\mathcal{F}), \beta = [\operatorname{supp} \mathcal{F}] \}.$$

In the example we have

$$0 \to I_{C_t^0 \cup C_t^1} \to [\mathcal{O}_X \xrightarrow{s} \mathcal{O}_{C_t^0} \oplus \mathcal{O}_{C_t^1}] \to 0$$

when $t \neq 0$ and the limit is

$$0 \to I_{C_0^0 \cup C_0^1} \to [\mathcal{O}_X \xrightarrow{s} \mathcal{O}_{C_0^0} \oplus \mathcal{O}_{C_0^1}] \to \mathcal{O}_p \to 0.$$

So the cokernel of the limit stable pair, is not flat but is a skyscraper sheaf.

Let $\mathcal{O}_X \to \mathcal{F}$ be a stable pair with $\operatorname{supp}(\mathcal{F}) = C$ a smooth curve. So $\mathcal{F}$ must be a line bundle on C with a non-trivial section; i.e. $\mathcal{F} = \mathcal{O}(D)$ for $D = \sum n_i p_i$ an effective divisor (figure 3.2).

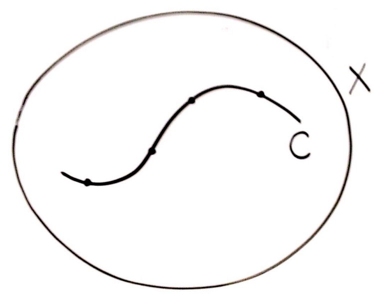


Figure 3.2: Points (given by divisor D) are now constrained to lie on the curve (C)

So we still have points and curves but the points are now constrained to lie on the curve. If $C \subset X$ is an isolated smooth curve of genus g then

$$PT_n(X, [C]) = \operatorname{Sym}^{n+g-1}(C)$$

where $n = 1 - g + |D|$. Comparing this to $I_n(X, \beta)$ we see that $PT_n(X, \beta)$ is typically smaller and nicer. We will see that the PT moduli space is the most efficient moduli space as far as curve counting is considered.

Recall that the bounded derived category $D^b(\mathcal{C}oh(X))$ is the category of bounded complexes

$$E^\bullet = \{ \cdots \to E^{-1} \to E^0 \to E^1 \to \cdots \}$$

where we localize the quasi-isomorphisms. The deformation theory of objects in this category is expressed in terms of trace free Hom, $\operatorname{Hom}_0(E^\bullet, F^\bullet)$.

Theorem 3.3 (Thomas)**.** *If we consider deformations of pairs $\mathcal{O} \xrightarrow{s} \mathcal{F}$ as objects $\{I^\bullet\}$ of the derived category then the moduli space is the same $PT_n(X, \beta)$ but the obstructions change. The space of deformations stays the same:*

$$\operatorname{Def}(\{I^\bullet\}) = \operatorname{Ext}^1_0(I^\bullet, I^\bullet)$$

but the obstructions change

$$\operatorname{Def}(\{I^\bullet\}) = \operatorname{Ext}^2_0(I^\bullet, I^\bullet).$$

Feb 28

A conclusion is that $PT_n(X, \beta)$ is locally given by

$$\kappa = df : \operatorname{Ext}^1_0(I^\bullet, I^\bullet) \to \operatorname{Ext}^2_0(I^\bullet, I^\bullet) \cong \operatorname{Ext}^1_1(I^\bullet, I^\bullet)^\vee$$

where $f : \mathrm{Ext}^1_0(I^\bullet, I^\bullet) \to \mathbb{C}$. Our previous machinery goes through and we have

$$N^{PT}_{n,\beta}(X) = \int_{[PT_n(X,\beta)]^{vir}} 1 = e_{vir}(PT_n(X,\beta))$$

where the right hand side is the Euler characteristic weighted by Behrend function. We have the analogous definitions

$$Z^{PT}_\beta(X) = \sum_n PT_{n,\beta}(X)q^n$$
$$Z^{PT}(X) = \sum_\beta Z^{PD}_\beta(X)v^\beta.$$

Note that from the definitions if $\beta = 0$ the only stable pair $\mathcal{O}_X \xrightarrow{0} 0$ and $Z^{PT}_0(X) = 0$.

Example 3.4. If $C \subset X$ is a smooth curve of genus g, a stable pair supported in C must be $\mathcal{O}_X \xrightarrow{s} \mathcal{O}(D)$ where $D = \sum n_i p_i$ is an effective divisor ($n_i > 0$), and $n = \chi(\mathcal{O}(D)) = \deg(D) + 1 - g$. Pairs supported on C of euler characteristic n give the moduli space $\mathrm{Sym}^{n+g-1}(C)$.

Example 3.5. Let $X = \mathrm{tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1)$ be the local $\mathbb{P}^1$. Then

$$PT_n(X, [\mathbb{P}^1]) = \mathrm{Sym}^{n-1}(\mathbb{P}^1) = \mathbb{P}^{n-1}$$

and $N^{PT}_{n,[\mathbb{P}^1]}(X) = (-1)^{n-1}n$. The partition function is

$$\begin{aligned} Z^{PT}_{[\mathbb{P}^1]}(X) &= \sum_{n=1}^\infty (-1)^{n-1} n q^n \\ &= -\sum_{n=1}^\infty n(-q)^n = -\frac{-q}{(1-(-q))^2} \\ &= \frac{q}{(1+q)^2}. \end{aligned}$$

Which we recall to be the same as $Z^{PT}_{[\mathbb{P}^1]}(X)'$.

Now the goal of this theory is achieved by the conjecture of Pandharipande-Thomas which was proved first by Todo and later by Bridgeland using machinery of Joyce-Song and Kontsevich-Soiblemann:

Theorem 3.6 (Todo, Bridgeland)**.**

$$Z^{PT}_\beta(X) = \frac{Z^{DT}_\beta(X)}{Z^{DT}_0(X)}.$$

Example 3.7. Let $N \to C$ be a rank 2 bundle on a smooth curve of genus g with $\wedge^2 N \cong K_C$ and such that $H^0(C, N) = 0$ and consider the local curve $X = \mathrm{tot}(N)$. Then the zero section $C \hookrightarrow N$ is rigid and

$$PT_n(X, [C]) = \{\mathcal{O}_X \xrightarrow{s} L, L = \mathcal{O}(D), |D| = n + g - 1\} = \mathrm{Sym}^{n+g-1}(C)$$

which is smooth of dimension $n + g - 1$. And

$$N^{PT}_{n,[C]}(X) = (-1)^{n+g-1} e(\mathrm{Sym}^{n+g-1}(C)).$$

Then there is a theorem due to MacDonald in topology that for any topological space Y, we have

$$\sum_{k=0}^{\infty} e(\mathrm{Sym}^k(Y))y^k = \frac{1}{(1-y)^{e(Y)}}.$$

So we can compute

$$\begin{aligned} Z^{PT}_{[C]}(X) &= \sum_{n=1-g}^{\infty} (-1)^{n+g-1} e(\mathrm{Sym}^{n+g-1}(C))q^n \\ &= q^{1-g}\sum_{k=0}^{\infty} e(\mathrm{Sym}^k(C))(-q)^k = q^{1-g}(1+q)^{2g-2} \\ &= (q^{-1/2}+q^{1/2})^{2g-2} = (q^{-1}+2+q)^{g-1}. \end{aligned}$$

We may now check the correspondence here by change of variable $q = -e^{i\lambda}$:

$$\begin{aligned} Z^{PT}_{[C]}(X) &= (2-e^{-i\lambda}-e^{i\lambda})^{g-1} \\ &= (2-2\cos(\lambda))^{g-1} = (4\sin^2(\frac{\lambda}{2}))^{g-1} = (2\sin(\frac{\lambda}{2}))^{2g-2}. \end{aligned}$$

So

$$\begin{aligned} Z^{GW}_{[C]}(X)' &= (\lambda - \frac{2}{6}\frac{\lambda^3}{8} + \cdots)^{2g-2} \\ &= \lambda^{2g-2}(1-\frac{\lambda^2}{24}+\cdots)^{2g-2} \\ &= 1.\lambda^{2g-2} - \frac{g-1}{12}\lambda^{2g} + \cdots . \end{aligned}$$

Recall that these invariants a priori count the possibility disconnected Gromov-Witten invariants. But because of geometry of C there are no maps of disconnected curves, hence these are actually the connected Gromov-Witten invariants:

$$Z^{GW}_{[C]}(X)' = F^{GW}_{[C]}(X)'.$$

And the Gupakumar-Vafa conjecture is saying that

$$\begin{aligned} F^{GW}(X)' &= \sum_{\beta=0, g\geq 0} N^{GW}_{g,\beta}(X)\lambda^{2g-2}v^{\beta} \\ &= \sum_{\beta\neq 0, n\geq 0} n_{n,\beta}(X)\sum_{d=1}^{\infty}\frac{1}{d}(2\sin(\frac{d\lambda}{2}))^{2g-2}v^{d\beta}. \end{aligned}$$

And $F^{GW}_{[C]}(X)'$ is the $v^{[C]}$ term of above. So

$$n_{h,[C]} = \begin{cases} 1 & h = g \\ 0 & h \neq g. \end{cases}$$

But what are $n_{h,d[C]}$s? We yet have to answer!

Chapter 4

Gupakumar-Vafa Invariants

Mar 3

So far we have been studying three cornerstones of the general picture 4.1. Physics interpretation suggests that Gupakumar-Vafa invariants must be the most fundamental invariants among the ones we are investigation. But the issue with them is that unlike the other invariants it is hard to invent a rigorous mathematical framework for them. One idea is to try to use the conjectural correspondences between other invariants and potential Gupakumar-Vafa invariants as means of definition of them.

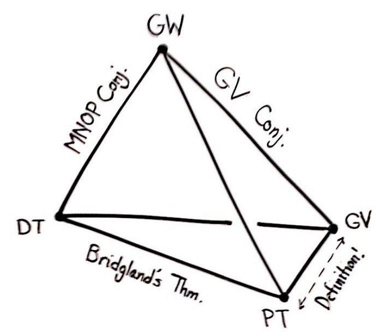


Figure 4.1: Curve-counting invariants and their correspondences

4.1 BPS conjecture

Recall that

$$\begin{aligned} Z^{PT} &= \sum_{\beta} Z^{PT}_{\beta}(q)v^{\beta} \\ &= \sum_{\beta} Z^{GW}_{\beta}(\lambda)'v^{\beta} \\ &= \exp\left(\sum_{\beta\neq 0} F^{GW}_{\beta}(\lambda)v^{\beta}\right) \\ &= \exp\left(\sum_{\beta\neq 0}\sum_{g\geq 0} N^{GW}_{g,\beta}(X)\lambda^{2g-2}v^{\beta}\right) \end{aligned}$$

which results in view of the multiple-cover formula

$$\begin{aligned} Z^{PT} &= \exp\left(\sum_{\beta\neq 0, g\geq 0} n_{g,\beta}(X) \sum_{d>0} \frac{1}{d}\left(2\sin\left(\frac{d\lambda}{2}\right)\right)^{2g-2} v^{d\beta}\right) \\ &= \exp\left(\sum_{g\geq 0,\beta\neq 0} n_{g,\beta} \sum_{d>0} \frac{1}{d}\left(\frac{1}{i}\left(e^{\frac{id\lambda}{2}} - e^{\frac{-id\lambda}{2}}\right)\right)^{2g-2} v^{d\beta}\right) \\ &= \exp\left(\sum_{g\geq 0,\beta\neq 0} n_{g,\beta} \sum_{d>0} \frac{1}{d}\left(-e^{-id\lambda}(e^{id\lambda}-1)^2\right)^{g-1} v^{d\beta}\right) \\ &= \exp\left(\sum_{g\geq 0,\beta\neq 0} n_{g,\beta} \sum_{d>0} \frac{(-1)^{g-1}}{d}\left((-q)^{-d}(1-(-q)^d)^2\right)^{g-1} v^{d\beta}\right) \end{aligned}$$

So we have the following formal identity between these invariants

$$\sum_{\beta,n} N^{PT}_{n,\beta}(X) q^n v^\beta = \exp\left(\sum_{g\geq 0,\beta\neq 0} n_{g,\beta} \sum_{d>0} \frac{(-1)^{g-1}}{d}\left((-q)^{-d}(1-(-q)^d)^2\right)^{g-1} v^{d\beta}\right) \tag{4.1}$$

So one can try to take this as the definition of Gupakumar-Vafa invariant. In fact, any series of the form of the left hand side, where the coefficients of v^β are Laurent series in q with integer coefficients, can be written in the form of the right hand side if we sum $g = -\infty, \cdots, c(\beta)$ with $n_{g,\beta} \in \mathbb{Z}$.

So the upshot of this story is that we can use 4.1 with allowing g to be negative as well, to define $n_{g,\beta}(X)$ in terms of $N^{PT}_{g,\beta}$. Then $n_{g,\beta} \in \mathbb{Z}$ is automatic. But the conjecture now is that

$$\boxed{n_{g,\beta} = 0, \forall g < 0} \tag{4.2}$$

BPS Conjecture

A curve class $\beta \in H_2(X)$ is is said to be irreduicble if $\beta \neq \beta_1 + \beta_2$ with β_i's being effective.

We can look at v^β in for irreducible classes (hence $v^\beta \neq v^{\beta_1} v^{\beta_2}$) so

$$\begin{aligned} Z^{PT}_\beta(q) &= \sum_g n_{g,\beta}(-1)^{g-1}\left(-\frac{(1+q)^2}{q}\right)^{g-1} \\ &= \sum_g n_{g,\beta} q^{1-g}(1+q)^{2g-2} \\ &= \sum_g n_{g,\beta}(q^{-1}+2+q)^{g-2} \end{aligned}$$

Any Laurent series in q can be written in the form $\sum_{r\leq C} a_r q^{1-r}(1+q)^{2r-2}$ since

$$q^{1-2}(q+1)^{2r-2} = q^{1-2} + \text{higher order terms}\ .$$

We can prove the BPS-conjecture in this case (for irreducible β). We wish to show that for an irreducible class β we have

$$Z_{\beta}^{PT} = \sum_{r} N_{n,\beta}^{PT} q^n = \sum_{g=0}^{C(\beta)} n_{g,\beta} q^{1-g}(1+q)^{2g-2}.$$

For $g \geq 1$ the Laurent polynomials

$$q^{1-g}(1+q)^{2g-2} = (q^{-1}+2+q)^{g-1}$$

are $1, q^{-1}+2+q, q^{-2}+4q^{-1}, 6+4q+q^2, \cdots$.

For $g = 0$, we have

$$\frac{1}{(1+q)^2} = q - 2q^2 + eq^3 + \cdots = \sum(-q)^{n-1} n q^n.$$

So we want to prove that

$$N_{n,\beta}^{PT} = N_{-n,\beta}^{PT} + c(-1)^{n-1} n, \quad c = n_{0,\beta}$$

We will see that the proof is a consequence of Serre duality. Note that when we prove this we also prove a part of the MNOP conjecture which is that we are allowed to make that substitution.

Let $\mathcal{M}_n(X,\beta)$ be the moduli space of stable, pure sheaves $\mathcal{F}$ with $\chi(\mathcal{F}) = n$ and such that $[\mathrm{supp}(\mathcal{F})] = \beta$. This space is a scheme by results of Simpson.

Let β be irreducible, then if $\mathcal{O} \xrightarrow{s} \mathcal{F}$ is a PT-pair then $\mathcal{F}$ is stable. The map

$$\Phi_n : PT_n(X,\beta) \to M_n(X,\beta)$$

mapping $(\mathcal{F}, s) \mapsto \mathcal{F}$ is therefore well-defined. And the fibers of this map are $\Phi_n^{-1}([\mathcal{F}]) = \mathbb{P}(H^0(X,\mathcal{F}))$. We stratify $M_n(X,\beta)$ by the dimension of $H^0(X,\mathcal{F})$:

$$M_n(X,\beta) = \bigsqcup_{\alpha} V_\alpha.$$

Then $\Phi^{-1}{}_n(V_\alpha) \to V_\alpha$ is a fiber bundle with fibers $\mathbb{P}^{\alpha-1}$so $e(PT_n(X,\beta)) = \sum_\alpha \alpha e(V_\alpha)$. So the Behrend function is compatible with Φ_n.

Lemma 4.1 (PT). *We have* $\nu_{PT_n(X,\beta)} = (-1)^{n-1}\Phi_n^*(\nu_{M_n(X,\beta)})$. *Thus we will have*

$$N_{n,\beta}^{PT}(X) = e_{vir}(PT_n(X,\eta)) = \sum_{\alpha} \alpha(-1)^{n-1} e_{vir}(V_\alpha).$$

There is an isomorphism $M_n(X,\beta) \xrightarrow{\cong} M_{-n}(X,\beta)$ via $\mathcal{F} \mapsto \mathcal{F}^\vee$.

$$\begin{tikzcd}
PT_n(X,\beta) \arrow[d, "\Phi_n"'] & PT_{-n}(X,\beta) \arrow[d, "\Phi_{-n}"] \\
\mathcal{M}_n(X,\beta) \arrow[r] & \mathcal{M}_{-n}(X,\beta)
\end{tikzcd}$$

with fibers $\Phi^{-1}{}_{-n}(V_\alpha) \to V_\alpha$ being isomorphic to $\mathbb{P}^{\alpha-n-1}$.

Also note that

$$H^0(X, \mathcal{F}^\vee) = H^0(X, R^\bullet \mathcal{H}om(\mathcal{F}, \mathcal{O}))$$

where a priori $R^\bullet \mathcal{H}om(\mathcal{F}, \mathcal{O})$ is a complex of sheaves with cohomology $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O})$. But because $\operatorname{codim}(\operatorname{supp} \mathcal{F}) = 2$ we have $\mathcal{E}xt^{i<2} = 0$ and since $\mathcal{F}$ is pure we have $\mathcal{E}xt^3 = 0$. Therefore

$$H^0(X, \mathcal{F}^\vee) = H^0(R, \mathcal{E}xt^2(\mathcal{F}, \mathcal{O})) = \operatorname{Ext}^2(\mathcal{F}, \mathcal{O}) = H^1(X, \mathcal{F})^\vee$$

by Serre duality on Calabi-Yau threefolds.

Of course $n = h^0(\mathcal{F}) - h^1(\mathcal{F}) = h^0(\mathcal{F}) - h^0(\mathcal{F}^\vee)$. ... So we have

$$e_{vir}(PT_{-n}(X, \beta)) = \sum_\alpha (\alpha - n)(-1)^{n-1} e_{vir}(V_\alpha)$$

and hence

$$\begin{aligned} N^{PT}_{n,\beta} - N^{PT}_{-n,\beta} &= \sum_\alpha \alpha(-1)^{n-1} e_{vir}(V_\alpha) - (\alpha - n)(-1)^{-n+1} e_{vir}(V_\alpha) \\ &= (-1)^{n-1} n \sum_\alpha e_{vir} V_\alpha = (-1)^{n-1} n e_{vir}(M_n(X, \beta)). \end{aligned}$$

Finally $e_{vir}(M_n(X, \beta) = e_{vir}(M_{n+1}(X, \beta))$. This is true because locally the two spaces $M_n(X, \beta)$ and $M_{n+1}(X, \beta)$ are isomorphic via $[\mathcal{F}] \mapsto [\mathcal{F}(D)]$ for some $D.\beta = 1$. But such line bundle D exists locally by some technical details.

Conjecture 4.2 (Katz)**.** *For any β,*

$$n_{0,\beta}(X) = e_{vir}(M_1(X, \beta))$$

where the right hand moduli, is the moduli space of stable sheaves with euler characteristic one.

Chapter 5

Generalized Donaldson-Thomas Invariants

Mar 5

Our motivation for embarking on this topic is Bridgeland's proof of DT/PT correspondence through techniques of Wall-crossing and Hall algebras and also Li-Kiem's geometric definition of Gupakumar-Vafa invariants. Recall that we resorted to defining GV-invariants using other invariants and saw that finally in genus zero we can get a definition using Donaldson-Thomas type invariants, which is a shame because somehow Gupakumar-Vafa invariants are supposedly the most fundamental of our invariants; one's that are closest to actual curve-counting.

What are the basic ingredients for the construction of Donaldson-Thomas (Pandharipande-Thomas) invariants:

1. $\mathcal{M}$ is some moduli space of coherent sheaves (with objects in $D^b(\mathcal{C}oh(X))$) on a Calabi-Yau threefold X, and $\mathcal{M}$ is a finite type scheme.
2. Deformation theory is such that the moduli space locally at $[E] \in M$ is modelled on $\{df = 0\} \subset \mathrm{Def}(E)$ where the potential function is $f : \mathrm{Def}(E) \to \mathbb{C}$.
3. Behrend function $\nu_{\mathcal{M}} : \mathcal{M} \to \mathbb{Z}$ defined via
$$\nu_{\mathcal{M}}([E]) = (-1)^{\dim \mathrm{Def}(E)}(1 - e(MF_f(E))).$$
4. Numerical invairants:
$$e_{vir}(\mathcal{M}) := \sum_{k \in \mathfrak{Z}} k e(\nu_{\mathcal{M}}^{-1}(k)).$$

What did we really use here to get (1)–(4)? Property (2) is a categorical property of $\mathcal{C}oh(X)$, or more generally that of $D^b(\mathcal{C}oh(X))$. So suppose that $\mathscr{C}$ is an abelian category with a notion of families of objects in $\mathscr{C}$ (for $\mathcal{C} = \mathcal{C}oh(X)$ this is flat families). The infinitesimal deformations of $E \in Ob(\mathscr{C})$ are classified by $\mathrm{Ext}^1(E, E)$ and the obstructions are given by $\mathrm{Ext}^2(E, E)$ and the moduli stack is locally given by

$$\mathrm{Ext}^1(E, E) \xrightarrow{\kappa} \mathrm{Ext}^2(E, E)$$

and the moduli stack of objects in this category is locally expressed via $\{\kappa = 0\}/\operatorname{Aut}(E)$. If $\mathrm{Ext}^1(E, E) \cong \mathrm{Ext}^2(E, E)^\vee$ then $\kappa = df$.

Definition 5.1. An abelian category $\mathscr{C}$ is called Calabi-Yau of dimension n if

$$\mathrm{Ext}^i(E, F) \cong \mathrm{Ext}^{n-i}(F, E)^\vee$$

for all objects E and F in $\mathscr{C}$, and the moduli stack of objects in $\mathscr{C}$ forms an algebraic stack with stabilizers which are affine algebraic groups.

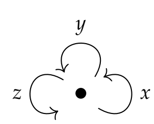
Any CY3 category satisfies property (2). For purposes of our course,

Remark 5.2. Recall that such stacks are locally quotient stacks $[X/G]$ with X a scheme and G is an affine algebraic group actin on X. In the stack world the quotient is "free" even if the action is not. If X is a manifold and G is finite the $[X/G]$ is an orbifold so we may think of these as generalization of orbifolds.

A theory of Donaldson-Thomas invariants for CY3 categories was predominantly invented by Kontsevich-Soibelmann and Joyce-Song. There are mainly two examples in which this goes through:

1. $\mathscr{C} = \mathcal{C}oh(X)$ as the abelian category of coherent sheaves on a Calabi-Yau threefold (that of Joyce-Song) and $D^b(\mathcal{C}oh(X))$ specifically in framework of Kontsevich-Soiblemann;
2. $\mathscr{C} = \mathcal{R}ep(Q, w)$, the category of representations of a quiver Q with a superpotential w .
3. $\mathscr{C}$ the flat complex-bundles on real oriented 3-manifold.

There are some simple categories that can be interpreted as all three above. For example if $\mathscr{C} = \mathcal{C}oh(\mathbb{C}^3)$ is the category of sheaves with compact support on $\mathbb{C}^3$, then this category is also isomorphic to $\mathcal{R}ep(Q, xyz - xzy)$ where Q is



Also $\mathscr{C}$ is the category of flat bundles on $T^3 = S^1 \times S^1 \times S^1$, or equivalently the category of $\mathbb{C}[x, y, z]$-module or that or triples of commuting matrices. Recall that in this case the superpotential is given via

$$f : (M_{n\times n}(\mathbb{C}))^3 \to \mathbb{C}$$

mapping $(X, Y, Z) \to \mathrm{tr}(X[Y, Z])$. The condition of $df = 0$ is equivalent to $[X, Y] = [Y, Z] = [X, Z] = 0$.

Recall that in ordinary (curve counting) Donaldson-Thomas theory we fixed numerical invariants n, β. We index components of the moduli space $\mathfrak{M}$, of objects in $\mathscr{C}$ by $\alpha \in K(\mathscr{C})$. So $\mathfrak{M}_\alpha$ is the moduli space of objects E with $[E] = \alpha \in K(\mathscr{C})$. $\mathcal{M}_\alpha$ typically fails to abe a scheme, it is a stack locally of the form $[\{df = 0\}/\operatorname{Aut} E]$ where $f : \operatorname{Ext}^1(E, E) \to \mathbb{C}$.

We can still define $\nu_{\mathfrak{M}_\alpha}$. Basic property of Behrend function: If

$$\begin{tikzcd} F \arrow[r] & P \arrow[d] \\ & E \end{tikzcd}$$

where π is smooth then

$$\nu_p = \pi^*(\nu_B).(-1)^{\dim F}$$

is the family version of the fact $\nu_{smooth} = (-1)^{\dim X}$. In particular if G acts freely on X then

$$\begin{tikzcd} G \arrow[r] & X \arrow[d, "\pi"] \\ & X/G \end{tikzcd}$$

and $\nu_X = (-1)^{\dim G}\pi^*(X/G)$ for a stack quotient $[X/G]$. So for

we define $\nu_{[X/G]}$ so that the above holds.. But then property (4) above is not satisfied: For a Zariski locally trivial bundle

$$\begin{tikzcd} F \arrow[r] & p \arrow[d] \\ & B \end{tikzcd}$$

we must have $e(P) = e(F).e(B)$ and $e_{vir}(P) = e_{vir}(F).e_{vir}(B)$.

If we want to define $e_{vir}[X/G]$ we are forced to have $e_{vir}[X/G] = \frac{e_{vir}(X)}{e_{vir}(G)}$ where $e_{vir}(G)$ is zero unless $\dim G = 0$. So we can do one of the two things:

- We need to restrict to open substacks $\mathfrak{M}^\circ_\alpha \subset \mathfrak{M}_\alpha$which is an actual scheme (this involved choosing s stability condition); or,
- We could generalize our notion of Euler characteristic to take values in a ring where the generalized euler characteristic of a group G is invertible in the ring $K_0(\mathsf{Var}_\mathbb{C})$ the Grothendieck group of varieties (a.k.a. ring of motives or ring of motivic classes).

5.1 Grothendieck groups

Definition 5.3. The Grothendick group of varieties, $K_0(\mathsf{Var}_\mathbb{C})$, is defind to be the free abelian group generated by isomorphism classes of varieties with the scissor relation

$$[V] = [V \setminus Z] + [Z]$$

where $Z \subseteq V$ is a closed subvariety.

$K_0(\mathsf{Var}_{\mathbb{C}})$ is a ring under the cartesian product: $[V].[W] = [V \times W]$. Immediately we see that if

$$\begin{tikzcd} F \arrow[r] & P \arrow[d] \\ & B \end{tikzcd}$$

is a Zariski locally trivial fibration then $[P] = [F].[B]$. Note that varieties are by definition of finite type (which is needed for noetherian induction for proof of this fact).

A few distinguished elements of this ring are $[\emptyset] = 0$ which is the additive identity, the motivic class of a point $[pt] = 1$ which is the multiplicative identity. The class of affine line is known as the *Lefschetz* motive $\mathbb{L} = [\mathbb{C}]$. It is easy to see that

$$[\mathbb{P}^n] = \mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + 1.$$

There is a ring homomorphism $e : K_0(\mathsf{Var}/\mathbb{C}) \to \mathbb{Z}$ via $[V] \mapsto e(V)$. Note that $e(\mathbb{L}) = 1$. In fact there are more general ring homomorphisms as such:

$$p_t(.) : K_0(\mathsf{Var}/\mathbb{C}) \to \mathbb{Z}[t]$$

via $[V] \mapsto p_t(V)$ such that if V is a projective manifold then $p_t(V) = \sum_{k=0}^{2\dim V} \dim H_k(V, \mathbb{Q}) t^k$ is the Poincaré polynomial of V. And note that $p_t(\mathbb{L}) = t^2$. In world of varieties, p_t is called the Serré polynomial or sometimes "virtual Poincaré polynomial".

Example 5.4. Let $X = \mathrm{Bl}_C(\mathbb{P}^3)$ where C is a curve of genus g. Then $[X] = [\mathbb{P}^3 \setminus C] + [E]$ where E is the exceptional divisor $\mathbb{P}(N_{C/\mathbb{P}^3}) \to C$. One can check that the normal bundle $N_{C/\mathbb{P}^3}$ is Zariski locally trivial on C and hence so is the projectivization. The fibers of $E \to C$ are isomorphic to $\mathbb{P}^1$ so we get

$$[X] = [\mathbb{P}^3] - [C] + [\mathbb{P}^1].[C] = 1 + \mathbb{L} + \mathbb{L}^2 + \mathbb{L}^3 + \mathbb{L}.[C].$$

The Poincaré polynomial is hence

$$\begin{aligned} p_t(X) &= 1 + t^2 + t^4 + t^6 + t^2(1 + 2gt + g^2) \\ &= 1 + 2t^2 + 2gt^3 + 2t^4 + t^6. \end{aligned}$$

The coefficients are therefore the Betti numbers $h^0 = h^6 = 1, h^2 = h^4 = 2$ and $h^3 = 2g$.

Another easy computation is the motivic class of the general linear group:

Example 5.5. We have a fibration $\mathrm{Gl}_n(\mathbb{C}) \to \mathbb{C}^n \setminus \{0\}$ by mapping any matrix to its first column vector. The fiber of this map over vector

v_1, called $F^1_{v_1}$ has a Zariski locally trivial on 2nd column $F^1_{v_1} \to \mathbb{C}^n \setminus \operatorname{span}\{v_1\} = \mathbb{C}^n \setminus \mathbb{C}$ and we can keep going inductively to get

$$[\mathrm{GL}_n(\mathbb{C})] = \mathbb{L}^{\binom{n}{2}}(\mathbb{L}^n - 1)(\mathbb{L}^{n-1} - 1)\cdots(\mathbb{L} - 1)$$

and by notation $\mathbb{L}^{\binom{n}{2}}[n!]_{\mathbb{L}}$.

Note that GL_n is not a projective variety, and in fact the virtual Poincaré polynomial of it has many roots in 1. But there is a nice way of making projective varieties from them:

Example 5.6. Let $Gr(k,n)$ be the Grassmanian of k planes in $\mathbb{C}^n$. We have

$$\begin{aligned}[Gr(k,n)] &= \frac{[\mathrm{GL}_n]}{[\mathrm{GL}_k][\mathrm{GL}_{n-k}][\mathbb{C}^{k(n-k)}]} \\ &= \frac{\mathbb{L}^{\binom{n}{2}}[n!]_{\mathbb{L}}}{\mathbb{L}^{\binom{k}{2}}[k!]_{\mathbb{L}}\mathbb{L}^{\binom{n-k}{2}}[(n-k)!]_{\mathbb{L}}\mathbb{L}^{k(n-k)}} \\ &= \frac{[n!]_{\mathbb{L}}}{[k!]_{\mathbb{L}}[(n-k)!]_{\mathbb{L}}}\end{aligned}$$

because $\binom{n}{2} = \binom{k}{2} + \binom{n-k}{2} + k(n-k)$. By notation this is written as

$$\binom{n}{k}_{\mathbb{L}} = \frac{(\mathbb{L}^n - 1)(\mathbb{L}^{n-1} - 1)\cdots(\mathbb{L}^{n-k+1} - 1)}{(\mathbb{L}^k - 1)\cdots(\mathbb{L} - 1)}$$

Note that by substituting $\mathbb{L}$ with 1, we get the usual binomial polynomial $\binom{n}{k}$ though this can only be done through residues. The Poincaré polynomial is

$$p_t(Gr(k,n)) = \frac{(t^{2n} - 1)(t^{2n-2} - 1)\cdots(t^{2n-2k+2} - 1)}{(t^{2k} - 1)\cdots(t^2 - 1)}.$$

So for instance $[Gr(2,4)] = 1 + \mathbb{L} + 2\mathbb{L}^2 + \mathbb{L}^3 + \mathbb{L}^4$ and hence the Betti numbers are respectively 1, 0, 1, 0, 2, 0, 1, 0 and 1.

If we look at $K_0(\mathsf{Var}_{\mathbb{Z}}) \subset K_0(\mathsf{Var}_{\mathbb{C}})$ then we get homomorphisms by counting $\mathbb{F}_q$-points of the varieties:

$$\#\mathbb{F}_q : K_0(\mathsf{Var}_{\mathbb{Z}}) \to \mathbb{Z}.$$

This gives another way of finding Poincaré polynomial of varieties. Another ring homomorphism from $K_0(\mathsf{Var}/\mathbb{C}) \to \mathbb{Z}$ is the Serre polynomial, which in particular maps $\mathbb{L}$ to t^2. In next class we will define Grothendieck groups of other categories of spaces of our interest.

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Let $K_0(\mathsf{Sch}/\mathbb{C})$ be the free abelian group generated by isomorphism classes of schemes of finite type over $\mathbb{C}$ with scissor relations. Also $K_0(\mathsf{St}/\mathbb{C})$ is the free abelian group of isomorphism classes of stacks of finite type over $\mathbb{C}$ with their analogous scissor relations. We have

$$K_0(\mathsf{Var}/\mathbb{C}) \hookrightarrow K_0(\mathsf{Sch}/\mathbb{C}) \hookrightarrow K_0(\mathsf{St}/\mathbb{C}).$$

Lemma 5.7. *$K_0(\mathsf{Var}/\mathbb{C})$ is isomorphic to $K_0(\mathsf{Sch}/\mathbb{C})$.*

The idea is that if for instance we consider $X = \operatorname{Spec}\mathbb{C}[x]/(x^2)$, then X_{red} is a point. Let

$$[X] = [\mathbb{A}^1] - [\mathbb{A}^1 \setminus X] = [\mathbb{A}^1] - [\mathbb{A}^1 \setminus X_{red} = [X_{red}]$$

because the Grothendieck group can only see the topology of the underlying space.[1]

[1] This is because of Bridgeland's relations that identifies $[X]$ and $[Y]$ if there is a geometric bijection $f : X \to Y$ between them. Otherwise it is not true only under scissor relations.

5.1.1 Grothendieck group of stacks

We will assume that all stacks are locally of finite type and have affine stabilizers. In particular this implies that we can locally write them as quotient stacks $[X/G]$ for X a scheme of finite type, and $G \subset \mathrm{Gl}_n(\mathbb{C})$ an affine algebraic group.

Theorem 5.8. *We have isomorphisms*

$$\begin{aligned} K_0(\mathsf{St}/\mathbb{C}) &\cong K_0(\mathsf{Var}/\mathbb{C})[\mathrm{GL}_n(\mathbb{C}) : n = 1, 2, \cdots] \\ &\cong K_0(\mathsf{Var}/\mathbb{C})[\mathbb{L}^{-1}, (1-\mathbb{L}^n)^{-1} : n = 1, 2, \cdots] \end{aligned}$$

The idea of the proof is to find a stratification of $\mathfrak{X}$ such that each strata is of the form $[Y/G]$ where Y is a scheme and $G \subset \mathrm{Gl}_n(\mathbb{C})$. Then we have

$$[Y/G] = [(Y \times \mathrm{GL}_n / G)/\mathrm{Gl}_n] = \frac{[Y \times \mathrm{GL}_n / G]}{[\mathrm{Gl}_n]}.$$

Also note that $K_0(\mathsf{Var}/\mathbb{C})[\mathbb{L}^{-1}, (1-\mathbb{L}^n)^{-1} : n = 1, 2, \cdots]$ is a subring of the completion of $K_0(\mathsf{Var}/\mathbb{C})$ in $\mathbb{L}$-adic topology:

$$K_0(\mathsf{Var}/\mathbb{C})[\mathbb{L}^{-1}, (1-\mathbb{L}^n)^{-1} : n = 1, 2, \cdots] \subset K_0(\mathsf{Var}/\mathbb{C})[\![\mathbb{L}^{-1}]\!].$$

5.1.2 Relative Grothendieck groups and relative motives

Recall that our goal is to make sense of $e_{vir}(\mathfrak{M})$ and $\nu_{\mathfrak{M}}$ such that items (3) and (4) above hold. We need a motivic version of Milnor fiber for this matter, i.e. one that is defined in terms of motives rather than topological spaces. And in place of a constructible function like ν and a weighted euler characteristic with respect to it, we need a function valued in motives and a motivic version of the weighted euler characteristic.

The second part is addressed by relative motives. Let S be a variety over $\mathbb{C}$. Then $K_0(\mathsf{Var}/S)$ is defined as the free abelian group generated by S-varieties $[f : X \to S]$, with relation

$$[f : X \to S] = [f|_U : U \to S] + [f|_Z : Z \to S]$$

where $Z \subset X$ is closed subvariety and $U = X \setminus Z$ is the complement. Similarly one can defined the relative Grothendieck group of stacks $K_0(\mathsf{St}/\mathfrak{S})$ where $\mathfrak{S}$ can be a stack.

Given $\varphi : \mathfrak{S} \to \mathfrak{T}$ a map of stacks, we can defined the pushforward

$$\varphi_* : K_0(\mathsf{St}/\mathfrak{S}) \to K_0(\mathsf{St}/\mathfrak{T}), \quad \text{via}[f : \mathfrak{X} \to \mathfrak{S}] \mapsto [\varphi \circ f : \mathfrak{X} \to \mathfrak{T}]$$

and pullback

$$\varphi^* : K_0(\mathsf{St}/\mathfrak{T}) \to K_0(\mathsf{St}/\mathfrak{S}), \quad \text{via}[f : \mathfrak{X} \to \mathfrak{T}] \mapsto [\mathfrak{X} \times_T \mathfrak{S} \to \mathfrak{T}].$$

So we can think of $[X \to S] \in K_0(\mathsf{Var}/\mathbb{C})$ as a motive-valued function on S which to each S-point $i : S_0 \hookrightarrow S$ assigns

$$i^*([X \to S]) \in K_0(\mathsf{Var}/S_0) = K_0(\mathsf{Var})$$

we may then compose this with the function $K_0(\mathsf{Var}/S) \to K_0(\mathsf{Var}/\mathbb{C})$ via $[X \to S] \mapsto [X]$ to get a weighted euler characteristic: more specifically we get a constructible function

$$\nu : S \to \mathbb{Z} \quad \text{via} S_0 \mapsto e(f^{-1}(S_0)).$$

Then

$$e(S, \nu) = \sum_{k \in \mathbb{Z}} k e(\nu^{-1}(k)) = e([X]).$$

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5.2 *Hall algebra of stacks*

Let $\mathscr{C}$ be a CY3 category. Our favorite examples are $\mathscr{C} = \mathcal{C}oh(X)$ or the stack of cherent sheaves of dimension ≥ 1 on a Calabi-Yau threefold, $\mathscr{C} = \mathcal{C}oh_{\leq 1}(X)$. The Hall-algebra, $H(\mathscr{C})$, is additively the Grothendieck group, $K_0(\mathsf{St}/\mathfrak{C})$, of stacks relative to $\mathfrak{C}$ (i.e. by abuse of notation, the moduli stack of objects in $\mathscr{C}$). Recall that $K_0(\mathsf{St}/\mathfrak{C})$ consists of classes

$$[f : \mathfrak{X} \to \mathfrak{C}]$$

of stacks of finite type $\mathfrak{X}$ with the scissor relations

$$[f : \mathfrak{X} \to \mathfrak{C}] = [f|_{\mathfrak{Z}} : \mathfrak{Z} \to \mathfrak{C}] + [f|_{\mathfrak{X} \setminus \mathfrak{Z}} : \mathfrak{X} \setminus \mathfrak{Z} \to \mathscr{C}].$$

The Hall product

$$[\mathfrak{X} \to \mathfrak{C}] * [\mathfrak{Y} \to \mathfrak{C}] = [\mathfrak{Z} \to \mathfrak{C}]$$

is the $\mathfrak{C}$-stack $\mathfrak{Z}$, of triples (x, y, E) where x and y are objects in $\mathfrak{X}$ and $\mathfrak{Y}$ and E is an extension

$$0 \to f(x) \to E \to g(y) \to 0.$$

More categorically, let $\mathfrak{C}_{ext}$ be the stack of short exact sequences in $\mathscr{C}$. And we define maps to $\mathfrak{C}$ via

$$p_i : (0 \to E_1 \to E_2 \to E_3 \to 0) \mapsto E_i.$$

and the Hall product $* : H(\mathscr{C}) \otimes H(\mathscr{C}) \to H(\mathscr{C})$ is defined as $(p_2)_*(p_1 \times p_3)^*$ as in the following diagram

$$\begin{tikzcd}
\mathfrak{Z} \arrow[r] \arrow[d] & \mathfrak{C}_{ext} \arrow[r] \arrow[d] & \mathfrak{C} \\
\mathfrak{X} \times \mathfrak{Y} \arrow[r, "f \times g"'] & \mathfrak{C} \times \mathfrak{C} &
\end{tikzcd}$$

Remark 5.9. The multiplicative unit in $H(\mathscr{C})$ is $1 = [0 \to \mathscr{C}]$.

5.2.1 *Associativity*

Let $\mathfrak{C}_{flag}$ be the stack of flags $F_1 \subset F_2 \subset F_3$ of objects F_i in $\mathscr{C}$. Then the following diagram is easily seen to be cartesian:

$$\begin{tikzcd}
\mathfrak{C}_{flag} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{C}_{ext} \arrow[d] \\
\mathfrak{C}_{ext} \times \mathfrak{C} \arrow[r] & \mathfrak{C} \times \mathfrak{C}.
\end{tikzcd}$$

Here the upper horizontal map is

$$(F_1 \subset F_2 \subset F_3) \mapsto (0 \to F_2 \to F_3 \to F_3/F_2 \to 0)$$

and the left vertical map is

$$(F_1 \subset F_2 \subset F_3) \mapsto (F_1 \to F_2 \to F_2/F_1, F_3/F_2).$$

Then the composition $H(\mathscr{C}) \otimes H(\mathscr{C}) \otimes H(\mathscr{C}) \xrightarrow{*\otimes \mathrm{id}} H(\mathscr{C}) \otimes H(\mathscr{C}) \to *H(\mathscr{C})$ is induced by the diagram

$$\begin{tikzcd}
\mathfrak{C}_{flag} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{C}_{ext} \arrow[r] \arrow[d] & \mathfrak{C} \\
\mathfrak{C}_{ext} \times \mathfrak{C} \arrow[r] \arrow[d] & \mathfrak{C} \times \mathfrak{C} & \\
\mathfrak{C} \times \mathfrak{C}\mathfrak{C} & &
\end{tikzcd}$$

So $* \circ (* \otimes \mathrm{id}) : H(\mathscr{C}) \otimes H(\mathscr{C}) \otimes H(\mathscr{C}) \to H(\mathscr{C})$ is induced actually by

$$\begin{tikzcd}
\mathfrak{C}_{flag} \arrow[r] \arrow[d] & \mathfrak{C} \\
\mathfrak{C} \times \mathfrak{C} \times \mathfrak{C} &
\end{tikzcd}$$

where the vertical map is $F_1 \subset F_2 \subset F_3 \mapsto (F_1, F_2/F_1, F_3/F_2)$ and the horizontal arrow maps the same flag to F_3.

The observation is that the other composition $* \circ (\mathrm{id} \otimes *)$ is also induced by the same diagram. This proves the associativity. Also notice that this gives a means of thinking about n-fold Hall products.

5.2.2 *Equations in Hall algebras*

Fix $\mathscr{C} = \mathcal{C}oh_{\leq 1}(X)$ for a Calabi-Yau threefold X. Let $\mathscr{P} \subset \mathscr{C}$ be the subcategory of $\mathscr{P} = \mathcal{C}oh_0(X)$ of sheaves of dimension zero. Let $\mathscr{Q} \subset \mathscr{C}$ be the subcategory of pure sheaves of dimension one, i.e. for any objet Q in $\mathscr{Q}$ we have $\mathrm{Hom}(P, Q) = 0$ for all objects P of $\mathscr{P}$. In this case we say $\mathscr{Q} = \mathscr{P}^{\perp}$.

Note that $\mathscr{O}_X \xrightarrow{s} F$ is a PT stable pair if and only if F is an object of $\mathscr{Q}$ and $\mathrm{coker}(s)$ is an object of $\mathscr{P}$. The pair $(\mathscr{P}, \mathscr{Q})$ is called a *torsion pair*; that is to say any $E \in \mathscr{C}$ can be written uniquely as an extension

$$0 \to P \to E \to Q \to 0$$

for some P and Q respectively objects of $\mathscr{P}$ and $\mathscr{Q}$.

We are interested in the elements

$$1_{\mathfrak{X}} = [\mathfrak{X} \hookrightarrow \mathfrak{C}]$$

where $\mathfrak{X}$ is the moduli stack of objects in $\mathscr{X}$ for cases of substacks $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{C}$ of $\mathfrak{C}$. Note that strictly speaking these elements are not in $H(\mathscr{C})$ because they are not of finite type. There are two ways of overcoming this:

1. We can grade $H(\mathscr{C})$ by elements λ in K-theory of X, $K_0(X)$ and then $1_{\mathfrak{X}}$ is finite type over each component $\mathfrak{C}_\lambda$. The Hall product will be defined over $\oplus_\lambda H_\lambda$ and respects the grading;
2. We may think of these as formal sums of elements in $H(\mathscr{C})$, in fact as elements in a formal completion $\widehat{H(\mathscr{C})}$.

This gives an identity in the Hall algebras:

$$\boxed{1_{\mathfrak{C}} = 1_{\mathfrak{P}} * 1_{\mathfrak{Q}}.} \tag{5.1}$$

Another element we are interested in is

$$I = [I_\bullet(X, \bullet) \to \mathfrak{C}]$$

where $I_\bullet(X, \bullet) = \bigsqcup_{n,\beta} I_n(X, \beta)$ and maps $(\mathscr{O}_X \to C) \mapsto C$. Let $\mathfrak{C}(\mathscr{O})$ be the stack of pairs (F, s) where F is an object in $\mathscr{C}$ and $s : \mathscr{O}_X \to F$. Then $I_\bullet(X, \bullet)$ is an open substack of $\mathfrak{C}(\mathscr{O})$. Let $1_{\mathfrak{C}}^{\mathscr{O}}$ be the element $[\mathfrak{C}(\mathscr{O}) \to \mathfrak{C}]$ where the structure morphism is $[\mathscr{O}_X \to E] \mapsto E$.

The following identity

$$\boxed{1_{\mathfrak{C}}^{\mathscr{O}} = I * 1_{\mathfrak{C}}} \tag{5.2}$$

says that every map factors uniquely as a surjection followed by an injection.

Given $\mathcal{O}_X \xrightarrow{f} F$ in $\mathfrak{C}(\mathcal{O})$ we can uniquely construct a diagram

$$\begin{tikzcd} & & \mathcal{O}_X \arrow[dl, two heads] \arrow[d, "f"] & & \\ 0 \arrow[r] & \mathcal{O}_X / \ker F \arrow[r] & F \arrow[r] & E \arrow[r] & 0 \end{tikzcd}$$

representing an element in the stack $I * 1_{\mathfrak{C}}$.

Conversely given an element in $I * 1_{\mathfrak{C}}$, i.e.

$$\begin{tikzcd} & \mathcal{O}_X \arrow[d] \arrow[dr, dashed] & & & & \\ 0 \arrow[r] & \mathcal{O}_C \arrow[r] & F \arrow[r] & E \arrow[r] & E/F \arrow[r] & 0 \end{tikzcd}$$

we get $\mathcal{O}_X \xrightarrow{f} F$ by composition .

Notice that our Hall algebra is an algebra over $K_0(\mathsf{Var}/\mathbb{C})$ via the product

$$[S].[f : \mathfrak{X} \to \mathfrak{C}] = [X \times S \to \mathfrak{C}].$$

Also since

$$K_0(\mathsf{St}/\mathbb{C}) = K_0(\mathsf{Var}/\mathbb{C})[\mathbb{L}^{-1}, (1 - \mathbb{L}^n)^{-1}]$$

our Hall algebra is a $K_0(\mathsf{Var}/\mathbb{C})[\mathbb{L}^{-1}]$-algebra via the induced structure.

Let $H_{reg}(\mathscr{C}) \subset H(\mathscr{C})$ be the sub $K_0(\mathsf{Var}/\mathbb{C})[\mathbb{L}^{-1}]$-module generated by $[\mathfrak{X} \to \mathfrak{C}]$ where X is a scheme.

Lemma 5.10. *$H_{reg}(\mathscr{C}) \subset H(\mathscr{C})$ is a $K_0(\mathsf{Var}/\mathbb{C})[\mathbb{L}^{-1}]$-subalgebra.*

Proof. The content of proof is to show that when X and Y are stacks, the product $[X \to \mathfrak{C}] * [Y \to \mathfrak{C}]$ can be written in terms of schemes over $\mathfrak{C}$. Let $[Z \to \mathfrak{C}]$ denote this Hall product. Then Z is the stack of triples (x, y, E) for $x \in X$ and $y \in Y$ and extension

$$0 \to f(x) \to E \to g(y) \to 0.$$

So Z is locally written as $\widetilde{Z}/\mathbb{C}^n$ where $\widetilde{Z}$ is a scheme. In fact the structure of $\mathrm{Aut}(x, y, E)$ is $\mathrm{Aut}\, x \times \mathrm{Aut}\, y \rtimes \mathrm{Hom}(g(y), f(x))$:

$$\mathrm{Aut}(x, y, E) = \begin{pmatrix} \mathrm{Aut}\, x & \mathrm{Hom}(g(y), f(x)) \\ 0 & \mathrm{Aut}\, y \end{pmatrix}.$$

But when X and Y are schemes $\mathrm{Aut}\, x$ and $\mathrm{Aut}\, y$ are trivial. $\square$

5.3 Integration map

A naive integration map is

$$\Phi^{eul} : H_{reg}(\mathscr{C}) \to \mathbb{Z}[K(\mathscr{C})]$$

via $[f : X \to \mathscr{C}]$ mapping to $\sum_{\alpha \in K(\mathscr{C})} e(X_\alpha) q^\alpha$ where $X_\alpha = f^{-1}(\mathfrak{C}_\alpha)$ for all elements in K-theory, $\alpha = (0, 0, \beta, n)$ and using notation $q^\alpha = q^n v^\beta$ notice that this is close to our DT partition functions.

A better integration map will incorporate virtual euler characteristics:

$$\Phi : H_{reg}(\mathscr{C}) \to \mathbb{Z}[K(\mathscr{C})]$$

via $[f : X \to \mathscr{C}]$ mapping to $\sum_{\alpha \in K(\mathscr{C})} e(X_\alpha, f^* \nu_{\mathfrak{C}}) q^\alpha$

It turns out that for maps $f : I_n(X, \beta) \to \mathfrak{C}$ the pullback of Behrend function is

$$f^* \nu_{\mathfrak{C}} = (-1)^n \nu_{I_n(X,\beta)}.$$

So for $I = [I_\bullet(X, \bullet) \to \mathfrak{C}]$ we have

$$\Phi(I) = \sum_{n,\beta} (-1)^n e_{vir}(I_n(X, \beta)) q^n v^\beta = Z^{Dt}(-q, v).$$

Let analogously define $PT = [PT_\bullet(X, \bullet) \to \mathfrak{C}$ with structure map $[\mathscr{O}_X \to F] \mapsto F$. Not that PT also has same type of stratification by curve class and euler characteristic, $PT_\bullet(X, \bullet) = \sqcup_{n,\beta} PT_n(X, \beta)$.

Lemma 5.11. *We have another Hall algebra identity*

$$\boxed{1_{\mathfrak{Q}}^{\mathfrak{C}} = PT * 1_{\mathfrak{Q}}} \tag{5.3}$$

Proof. Given $\mathscr{O} \xrightarrow{f} E$ and object in $\mathfrak{Q}(\mathscr{O})$ we can construct the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{O}_X \arrow[r] \arrow[d, dashed] & \mathscr{O}_X \arrow[r] \arrow[d, "f"] & 0 \arrow[d] & \\
0 \arrow[r] & F \arrow[r] \arrow[d, dashed] & E \arrow[r, "\gamma"'] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[d, dashed] & \operatorname{coker} f \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

We can complete the diagram as shown by diagram chasing. Notice that $F = \ker \gamma$ so $\mathscr{O}_X \xrightarrow{s} F$ is a PT pair. Also since $\mathscr{Q}$ is extension free F is in $\mathscr{Q}$ because E and Q are. So P is in $\mathscr{P}$ proving one direction. The other direction of the claim follows analogously. $\square$

Lemma 5.12. *Yet another identity in the Hall algebra*

Proof. The right hand side is stack of objects

$$\begin{tikzcd}
& \mathscr{O}_X \arrow[d] & & \mathscr{O}_X \arrow[d] & \\
0 \arrow[r] & P \arrow[r] & E \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

and $\mathfrak{C}(\mathcal{O})$ is the stack of objects

$$\begin{tikzcd} & & \mathcal{O}_X \arrow[d] & & \\ 0 \arrow[r] & P \arrow[r] & E \arrow[r] & Q \arrow[r] & 0 \end{tikzcd}$$

where both are Zariski locally trivial fibrations over the stack of objects

$$\begin{tikzcd} & & & \mathcal{O}_X \arrow[d] & \\ 0 \arrow[r] & P \arrow[r] & E \arrow[r] & Q \arrow[r] & 0 \end{tikzcd}$$

The fiber of the first one is $H^0(P)$ and the fiber of second map is $\ker(H^0(E) \to H^0(Q))$ But $H^1(P) = 0$ in the long exact sequence

$$0 \to H^0(P) \to H^0(E) \to H^0(Q) \to H^1(P)$$

because P is supported on points. This is enough to prove the identity in the Hall algebra. $\square$

5.4 *Bridgeland's argument*

Starting from identity 5.2 and using identities 5.1 and 5.3 we get

$$I * 1_{\mathfrak{P}} * 1_{\mathfrak{Q}} = I_0 * 1_{\mathfrak{P}} * PT * 1_{\mathfrak{Q}}. \tag{5.4}$$

If we could prove that $1_{\mathfrak{Q}}$ and $1_{\mathfrak{P}}$ were invertible in Hall algebra. Then we would get

$$I = I_0 * 1_{\mathfrak{P}} * PT * 1_{\mathfrak{P}}^{-1}.$$

This is true and needs to be proven.

Now let's pretend we can apply Φ (which we cannot the way we have defined it). Then we get

$$\Phi(I) = \Phi(I_0)\Phi(1_{\mathfrak{P}})\Phi(PT).\Phi(1_{\mathfrak{P}})^{-1} = \Phi(I_0)\Phi(PT)$$

which translates to

$$Z^{DT}(-q,v) = Z^{DT}(-q,0).Z^{PT}(-q,v).$$

There are two approaches to defining $\Phi(1_{\mathfrak{P}})$:

1. Joyce-Song: Use stability to show that $1_{\mathfrak{P}} = \exp\left(\frac{\varepsilon}{\mathbb{L}-1}\right)$ for $\varepsilon \in H(\mathscr{C})$ a regular element. Then show that Φ is well-defined for $1_{\mathfrak{P}} * PT * 1_{\mathfrak{P}}^{-1}$.
2. Kontsevich-Soiblemann: enlarge $\mathbb{Z}[K(\mathscr{C})]$ to something over which we can take Φ of stacks (leading to motivic DT/PT correspondence).

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Recall that for any Calabi-Yau category $\mathscr{C}$ of cohomological dimension ≤ 1, we derived an equation in the Hall algebra $H(\mathscr{C})$ (i.e. equation 5.4) in terms of classes $\mathfrak{C}$-stacks of moduli of objects of the following subcategories:

- $\mathscr{I}_0$, ideal sheave of 0 dimensional subschemes;
- $\mathscr{P}\mathscr{T}$ of PT pairs;
- $\mathscr{I}$, of ideal sheaves of 1 dimensional subschemes;
- $\mathscr{P}$, all sheaves of dimension 0;
- $\mathscr{Q}$, pure sheaves of dimension 1.

But in order for our DT/PT program to go through we need to extend the integration map

$$\Phi : H_{\text{reg}}(\mathscr{C}) \to \mathbb{Z}[\![K(\mathscr{C})]\!]$$

in a way that integration makes sense in bigger subalgebra of $H(\mathscr{C})$. In fact we will attempt to define Φ on all of $H(\mathscr{C})$. We need to enlarge the coefficient ring of range of Φ to some $R[\![K(\mathscr{C})]\!]$. In fact as we will see R will be a sopped up version of $K_0(\mathsf{St}/\mathbb{C}) = K_0(\mathsf{Var}/\mathbb{C})[\mathbb{L}^{-1}, (1-\mathbb{L}^k)^{-1} : k \geq 1]$.

Recall that if E is an object of $\mathscr{C}$ and $\mathscr{C}$ is locally near E given by $\{df = 0\}/\operatorname{Aut}(E)$ where $f : \operatorname{Ext}^1(E,E) \to \mathbb{C}$ then $\nu_{\mathfrak{C}}(E)$ can be expressed as

$$\nu_{\mathfrak{C}}(E) = \nu_{\{dt=0\}}/\nu_{\operatorname{Aut}(E)} = (-1)^{-\dim \operatorname{Ext}^0(E,E)}\nu_{\{df=0\}}(E)$$

in fact such identity is true for Behrend functions even if this quotient is not Zariski locally trivial. We can write this now as

$$\nu_{\mathfrak{C}}(E) = (-1)^{-\dim \operatorname{Ext}^0(E,E)+\dim \operatorname{Ext}^1(E,E)}(1 - e(MF_f(0))).$$

We want as assignment $E \mapsto w(E) \in \operatorname{Mot}$ (where Mot is the ring of motivic measures) such that $e(W(E)) = \nu_C(E)$. Here we have

$$\operatorname{Mot} = K_0^{\widehat{\mu}}(\mathsf{St}/\mathbb{C})[\mathbb{L}^{1/2}] = K_0^{\widehat{\mu}}(\mathsf{Var}/\mathbb{C})[\mathbb{L}^{-1}, (1-\mathbb{L}^n)^{-1}, \mathbb{L}^{1/2}]_{n\geq 1}.$$

For $w \in \operatorname{Mot}$ we can define euler characteristic of $e(w)$ (could be ∞) by assigning $e(\mathbb{L}^{1/2}) = -1$. The motivic version of Milnor fiber appears in the definition

$$w(E) := (\mathbb{L}^{1/2})^{-\dim \operatorname{Ext}^0(E,E)+\dim \operatorname{Ext}^1(E,E)}(1 - MF_f),$$

where MF_f here is the *motivic Milnor fiber*.

With some work we will see that $E \mapsto w(E)$ comes form an element $W \to \mathfrak{C}] \in K_0(\mathsf{St}/\mathbb{C})[\mathbb{L}^{1/2}]$, such that by pulling back along $i : \{E\} \hookrightarrow \mathfrak{C}$, we get

$$i^*[W \to \mathfrak{C}] = w(E).$$

So we need some sort of compatibility that guarantees this. We then define

$$\Phi^{\mathrm{mot}}([X \to \mathfrak{C}]) := \sum_{\alpha} [X_\alpha \times_{\mathfrak{C}} W] q^\alpha \in \mathrm{Mot}[\![K(\mathscr{C})]\!].$$

And all this happens to work because of the following

Theorem 5.13 (Kontsevich-Soiblemann)**.** *$\Phi^{\mathrm{mot}} : H(\mathscr{C}) \to \mathrm{Mot}[\![K(\mathscr{C})]\!]$ is a homomorphism through which $\Phi : H_{\mathrm{reg}} \to \mathbb{Z}[\![K(\mathscr{C})]\!]$ factors.*

5.5 Motivic Milnor fiber

Let $f : X \to \mathbb{C}$ be an analytic or algebraic functino on smooth variety X of dimension d. Assume that $\mathrm{crit}(f) = \{df = 0\} \subset X_0$.

Definition 5.14. Let $x_0 \in X_0$. Then the Milnor fiber of f at x_0 is defined as $MF_f(x_0) = f^{-1}(\delta) \cap B_\varepsilon(x_0)$ when $0 < \delta \ll \varepsilon \ll 1$.

Note that $f^{-1}(\{|y| = \delta\})$ is a fibration over S^1 with fibers diffeomorphic to $MF_f(x_0)$. This provides us with a monodromy action on the Milnor fiber. The cohomology $H^*(MF_f(x_0))$ now is equipped with a monodromy action (of finite order) and so we can view it as an action of $\mu_N = \{z^N = 1\} \subset \mathbb{C}^\times$ on $H^*(MF_f, \mathbb{Z})$ for some N. The jargon is that $e(MF_f(x_0))$ is called the *Milnor number* of f and the elements of $H^*(MF_f, \mathbb{Z})$ are called *nearby cycles*.

Let $\widehat{\mu} = \operatorname{proj\,lim} \mu_n$. We define $K_0(\widehat{\mu})(\mathsf{Var}/X_0)$ as the free abelian group generated by isomorphism classes of varieties over X_0 with a good action[2] of $\widehat{\mu}$ equivariant with respect tot the trivial action on X_0 moduli the relations:

[2] A good action of $\widehat{\mu}$ on V is one that factors through some μ_N for finite N and all μ_N-orbits are contained in an affine open set.

$$[V \to X_0] = [V \smallsetminus Z \to X_0] + [Z \to X_0]$$

for any $Z \subset V$, a $\widehat{\mu}$-invariant closed subvariety of Z, and

$$[X_0 \times W \to X_0] = [X_0 \times \mathbb{C}^n \to X_0]$$

where W is an n-dimension $\widehat{\mu}$ representation. $\mathbb{C}^n$ is the trivial n-dimensional representation. In other words

$$[X_0 \times W \to X_0, \widehat{\mu}] = \mathbb{L}^n, \quad \text{where} \quad \mathbb{L} = [\mathbb{A}^1 \times X_0 \to X_0].$$

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Now to do all of this in algebro-geometric setting, we use the idea of formal arcs. We will use arcs to probe the local geometry of X and f new $x_0 \in X$.

Definition 5.15. We define the order n arc-space at $x_0 \in X$, to be

$$\mathcal{L}_n(X) = \mathrm{Mor}(\mathrm{Spec}(\mathbb{C}[t]/t^{n+1}), X)$$

as an affine X-scheme, $\mathcal{L}_n(X) \to X$. Let $f : X \to \mathbb{C}$ be a morphism and $x_0 \in X_0$ live in the central fiber. We define the order n arcs in X which lie in X_0 to order $n-1$ and leaves the central fiber at order n to be the space

$$\mathcal{X}_{n,1} = \{\varphi \in \mathcal{L}_n(X) : f \circ \varphi = t^n\}.$$

Note that $\mathcal{X}_{n,1} \to X_0$ is also an X_0-scheme and it has a natural action of μ_n given by $\varphi(t) \mapsto \varphi(e^{2\pi i/n}t)$. So $[\mathcal{X}_{n,1}] \in \mathrm{Mot}_{X_0}$.

Theorem 5.16 (Denef, Loeser)**.** *The zeta function of arcs*

$$Z(T) = \sum_{n=1}^{\infty} [\mathcal{X}_{n,1}] \mathbb{L}^{-n \dim X} T^n \in \mathrm{Mot}_{X_0} [\![T]\!]$$

is a rational function in T provided that X is smooth.

Definition 5.17. The motivic Milnor fiber is finally defined as

$$MF_f = -\lim_{T\to\infty} Z(T) \in \mathrm{Mot}_{X_0}.$$

Example 5.18. Let $x_0 \in X_0$ be a smooth point of X_0. Let $(y_1, \cdots, y_d)$ be variables of coordinate ring locally at x_0 such that $f(y_1, \cdots, y_d) = y_1$. Then $\mathcal{X}_{n,1}$ is given by equations

$$a_0^i + a_1^i t + \cdots + a_n^i t^n, \quad i = 1, \cdots, d.$$

So $\mathcal{X}_{n,1} \to X_0$ is just the affine space $[X_0 \times \mathbb{C}^{(d-1)n} \to X_0]$. Then

$$\begin{aligned} Z(T) &= \sum_{n=1}^{\infty} [X_0 \to X_0] \mathbb{L}^{(d-1)n - nd} T^n \\ &= [X_0 \to X_0] \frac{\mathbb{L}^{-1}T}{1 - \mathbb{L}^{-1}T} \end{aligned}$$

As $T \to \infty$, we conclude that over the smooth part of X_0 we have

$$MF_f = [X_0 \to X_0]$$

i.e. the nearby fiber is isomorphic to central fiber over smooth points.

Definition 5.19. The motivic vanishing cycle is defined as

$$\varphi_f = [X_0] - MF_f \in \mathrm{Mot}_{\mathrm{crit}(f)} \subset \mathrm{Mot}_{X_0}.$$

Example 5.20. Consider $f(x) = x^2$ from affine line to itself. The space $\mathcal{X}_{n,1}$ are cut out by

$$\mathcal{X}_{n,1} = \{(z_0 + z_1 t + \cdots + z_n t^n)^2 = t^2 \mod t^{n+1}\}.$$

This determines the arc-spaces to be,

$$\begin{aligned} &\mathcal{L}_1 = \emptyset \\ &\mathcal{L}_2 = \{z_0 = 0, z_1^2 = 1, z_2 \in \mathbb{C}\} = [\text{two points}, \mu_2] \times \mathbb{L} \\ &\mathcal{L}_3 = \emptyset \\ &\mathcal{L}_4 = \{z_0 = z_1 = 0, z_2^2 = 1, z_3, z_4 \in \mathbb{C}\} = [\text{two points}, \mu_2] \times \mathbb{L} \\ &\quad\vdots \end{aligned}$$

So we have

$$\mathcal{L}_n = \begin{cases} \emptyset & n \text{ is odd} \\ \mu_2 \times \mathbb{L}^{n/2} & n \text{ is even} \end{cases}$$

$$\begin{aligned} Z(T) &= \sum_{a=1}^{\infty} \mu_2 \times \mathbb{L}^a \mathbb{L}^{-2a} T^{2a} \\ &= \mu_2 \sum_{a=1}^{\infty} (\mathbb{L}^{-1}T^2)^a \\ &= \mu_2 \frac{\mathbb{L}^{-1}T^2}{1-\mathbb{L}^{-1}T^2} \end{aligned}$$

Hence the motivic Milnor fiber is a copy of μ_2, which should not surprise us because the monodromy action of $x \mapsto x^2$ is by flipping two points. And the motivic vanishing cycle is $\varphi_f = 1 - \mu_2$.

Already we see that computing the motivic Milnor fiber can be cumbersome in most situations. Now we are going to discuss some tools that make this means easier.

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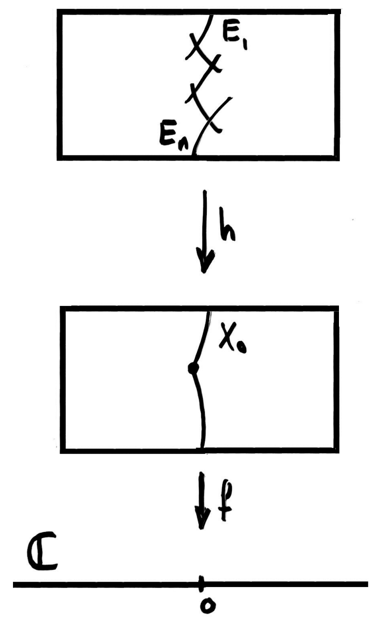

Figure 5.1: Resolution of singularities

5.5.1 *Computing the Milnor fibers using embedded resolutions*

Let $Y \xrightarrow{h} X$ be a smooth map and h is a proper resolution of singularities (hence being rational, dominant, and proper). We require h to be an isomorphism away from $\operatorname{crit}(f)$ and $h^{-1}(X_0)$ be a normal crossing divisor with except finitely many components $E_1, \cdots, E_n$.

Recall that the fact that h is a nomral crossing divisor means that all E_i are smooth and for any set of indices $I \subset \{1, \cdots, n\}$, $E_I := \cap_{i \in I} E_i$ is smooth of dimension $d - |I|$ where $d = \dim X$.

Using the notation $E_I^\circ = E_I \setminus E_I \cap \left(\cup_{j \notin I} E_j\right)$ we have

$$h^{-1}(X_0) = \sqcup_{\emptyset \neq I \subset \{1, \cdots, n\}} E_I^\circ.$$

Let $\sum n_i E_i$ be the pullback of the divisor $\{0\} \subset \mathbb{C}$ via $f \circ h$. We let

$$n_I = \gcd_{i \in I}(n_i).$$

Finally there exists an unramified cover of order n_I, $\widetilde{E_I^\circ} \to E_I^\circ$ so $[\widetilde{E_I^\circ}, \mu_{n_I}] \in \mathrm{Mot}_{X_0}$.

Theorem 5.21. *We can now compute the motivic Milnor fiber*

$$MF_f = \sum_{\emptyset \neq I \subset \{1, \cdots, n\}} (1 - \mathbb{L})^{|I|-1} [\widetilde{E_I^\circ}, \mu_{n_I}].$$

The covers $\widetilde{E_I^\circ} \to E_I^\circ$ are restrictions of (ramified covers) $\widetilde{E}_I \to E_I$ and $\widetilde{E}_I|_{E_{J \supset I}}$ is a disjoint union of a bunch of copies of $\widetilde{E}_J \to E_J$.

Example 5.22. Back to the case of $f(x) = x^2$ from $\mathbb{C}$ to $\mathbb{C}$, $X = Y$ in this case and E_0 is just a point and $n_0 = 2$. The covering $\widetilde{E_0} \to E_0$ is $\{\bullet, \bullet\} \to \{\bullet\}$. So we have $[\widetilde{E_0}] = \mu_@$ and $MF_f = \mu_2$.

Example 5.23. Given $X = \mathbb{C}^3 \to \mathbb{C}$ via $(x, y, z) \mapsto xyz$, the central fiber consists of the xy, yz and xz-planes (denoted as E_i^0 for $i = 1, 2, 3$) and the critical locus of f consists of the coordinate axis. So we have

$$MF_f = E_1^\circ + E_2^\circ + E_3^\circ + (1-\mathbb{L})(E_{12}^\circ + E_{13}^\circ + E_{23}^\circ) + (1-\mathbb{L})^2 E_{123} \in \mathrm{Mot}_{X_0}.$$

So we see that $\Phi_f = X_0 - MF_f$ is clearly supported on the critical locus. Let us compute the image of MF_f under the map $Mot_{X_0} \to Mot_{pt}$ (that forget maps to X_0).

$$\begin{aligned} MF_f &= 3([\mathbb{C}^\times \times \mathbb{C}^\times]) + (1-\mathbb{L})(3[\mathbb{C}^\times]) + (\mathbb{L}-1)^2 \\ &= (\mathbb{L}-1)^2(3-3+1) = [\mathbb{C}^\times \times \mathbb{C}^\times]. \end{aligned}$$

Example 5.24. Let $f : \mathbb{C}^n \to \mathbb{C}$ be given by $f(x_1, \cdots, x_n) = \sum x_i^2$. The singular fiber is a cone (so not normal crossings unless $n = 2$).

So we need to blow this up. And it turns out that the blow up at the origin suffices. We will use the notation

$$Q_{n-1} = \{x_0^2 + \cdots + x_n^2 = 0\} \subseteq \mathbb{P}^n$$

equipped with the $\mathbb{Z}/2$-action $x_0 \leftrightarrow -x_0$.

Let $Y = \mathrm{Bl}_0(\mathbb{C}^n)$ and $E_0 = \widetilde{X_0}$ be the proper transform of X_0. Let $E_1 \cong \mathbb{P}^{n-1}$ be the exceptional divisor of the blow up . The intersection of these subspaces is $E_{01} = Q_{n-2} \subset \mathbb{P}^{n-1}$.

$\widetilde{E_1^\circ} \to E_1^\circ$ is the restriction of the 2:1 map $\widetilde{E_1} \to E_1 \cong \mathbb{P}^{n-1}$ branched on $Q_{n-2} \subset \mathbb{P}^{n-1}$ (so in the notation of previous theorem, $n_0 = 1$, $n_1 = 2$ and $n_{01} = 1$). So $[\widetilde{E_1^\circ}, \mu_2] = [Q_{n-1}, \mu_2] - Q_{n-2}$. Now we compute the vanishing cycle

$$\begin{aligned} \Phi_f &= X_0 - (E_0^\circ + [\widetilde{E_1^\circ}, \mu_2] + (1-\mathbb{L})Q_{n-2}) \\ &= E_0^0 + 1 - E_0^0 - ([Q_{n-1}, \mu_2] - Q_{n-2}) - (1-\mathbb{L})Q_{n-2} \\ &= 1 + \mathbb{L}Q_{n-2} - [Q_{n-1}, \mu_2] \in K_0^{\hat{\mu}}(\mathsf{Var}/\mathbb{C}). \end{aligned}$$

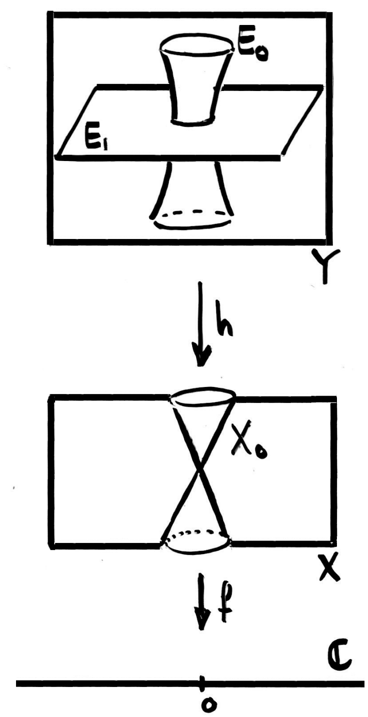

Figure 5.2: Resolution of the singularity $\sum_{i=1}^n x_i^2 = 0$

Let $p \in Q_n$ be a μ_2 invariant point. Then projection from p to $\mathbb{P}^n$ gives us a rational map $Q_n \dashrightarrow \mathbb{P}^n$. So we have

$$\begin{array}{ccc} \mathrm{Bl}_p Q_n & \xrightarrow{\cong} & \mathrm{Bl}_{Q_{n-2}}\mathbb{P}^n \\ \downarrow & & \downarrow \\ (Q_n, \mu_2) & & (\mathbb{P}^n, \mu_2) \end{array}$$

The right hand map contracts the quadric of lines in the tangent plane $T_p Q_n$. On $Q_{n-2} \subset \mathbb{P}(T, Q)$, μ_2 acts trivially.

So we have

$$[Q_n, \mu_2] - p + \mathbb{P}^{n-1} = [\mathbb{P}^n, \mu_2] - Q_{n-2} + \mathbb{P}^1.Q_{n-2}$$

which gives us a recursive formula,

$$[Q_n, \mu_2] = \mathbb{P}^n - \mathbb{P}^{n-1} + 1 + \mathbb{L}Q_{n-2} = \mathbb{L}^n + 1 + \mathbb{L}Q_{n-2}$$

starting in $Q_0 = \mu_2$ and $Q_1 = \mathbb{P}^1$. We solve the recursion and get

$$[Q_n, \mu_2] = \begin{cases} \mathbb{P}^n & n \text{ is odd} \\ \mathbb{P}^n + \mathbb{L}^{n/2}(\mu_2 - 1) & n \text{ is even} \end{cases}.$$

We conclude that the motivic vanishing cycle is

$$\Phi_f = [X_0] - MF_f = \begin{cases} \mathbb{L}^{n/2} & n \text{ is even} \\ \mathbb{L}^{n-1/2}(1-\mu_2) & n \text{ is odd} \end{cases}.$$

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5.6 *Motivic Thom-Sebastiani theorem*

Let $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ be two algebraic morphisms to $\mathbb{C}$, and define $f + g : X \times Y \to \mathbb{C}$ via $(x, y) \mapsto f(x) + g(y)$. We will define a convolution product on $\mathrm{Mot}_{X\times Y}$ such that the following theorem holds,

Theorem 5.25.

$$\Phi_{f+g} = (\pi_X^* \Phi_f) * (\pi_Y^* \Phi_g) \in \mathrm{Mor}_{X\times Y}.$$

We define the convolution product on $\mathrm{Mot}_{pt} = K_0^{\hat{\mu}}(\mathsf{Var}/\mathbb{C})/$ extra relations as follows.

First we note that to each variety X with a μ_d action, we may associate an X-bundle over $\mathbb{C}^\times$ with monodromy μ_d:

$$X \times_{\mu_d} \mathbb{C}^\times \to \mathbb{C}^\times$$

where the notation $X \times_{\mu_d} \mathbb{C}^\times$ is the quotient $X \times \mathbb{C}^\mu / \mu_d$ with the diagonal action. Conversely given such an object, we can recover (X, μ_d) by taking the fiber over 1.

Now, for X and Y with $\hat{\mu}$-action, we can find d such that μ_d acts on both spaces. We define $J(X, Y) \in \mathrm{Mot}$ as the fiber over 1 of $(X \times_{\mu_d} \mathbb{C}^\times) \times (Y \times_{\mu_d} \mathbb{C}^\times)$ equipped with the diagonal μ_d-action.

$$\begin{tikzcd} (X \times_{\mu_d} \mathbb{C}^\times) \times (Y \times_{\mu_d} \mathbb{C}^\times) \arrow[d] \arrow[dr] & \\ \mathbb{C}^\times \times \mathbb{C}^\times \arrow[r, "+"] & \mathbb{C} \end{tikzcd}$$

Definition 5.26. Give X, Y, and $J(X,Y)$ as above, we define the convolution product of X and Y by

$$X * Y = -\left(J(X,Y) - (\mathbb{L}-1)^{X \times Y}/\mu_d\right).$$

With this definition we get $X * 1 = X$.

Example 5.27. Let us abbreviate $[\bullet\bullet, \mu_2]$ as μ_2 and compute $\mu_2 * \mu_2$. Note that $\mu_2 \times_{\mu_2} \mathbb{C}^\times \to \mathbb{C}^\times$ is isomorphic to the degree two cover $\mathbb{C}^\times \to \mathbb{C}^\times$ via $u \mapsto u^2$. So

$$J(\mu_2, \mu_2) = \{(u,v) \in \mathbb{C}^\times \times \mathbb{C}^\times : u^2 + v^2 = 1\}$$

where the diagonal μ_2 action is $(u,v) \mapsto (-u,-v)$. But the subspace of $\mathbb{C} \times \mathbb{C}$ cut out by $u_2 + v^2 = 1$, is isomorphic to $[\mathbb{C}^\times, \mu_2]$ which tells us

$$J(\mu_2, \mu_2) = [\mathbb{C}^\times, \mu_2] - \mu_2 - \mu_2 = [\mathbb{C}, \mu_2] - 1 - 2\mu_2 = \mathbb{L} - 1 - 2\mu_2.$$

Therefore,

$$\begin{aligned} \mu_2 * \mu_2 &= -\left(\mathbb{L} - 1 - 2\mu_2 - (\mathbb{L}-1)^{(\bullet,\bullet)\times(\bullet,\bullet)}/\mu_2\right) \\ &= \mathbb{L} - 1 + 2\mu_2 \end{aligned}$$

Corollary 5.28. *This suggests that we may impose the following relation in the ring of motives:*

$$(1-\mu_2) = \mathbb{L}^{\frac{1}{2}} \in \mathrm{Mot}$$

because $(1-\mu_2) * (1-\mu_2) = 1 - 2\mu_2 + \mathbb{L} - 1 + 2\mu_2 = \mathbb{L}$.

With this introduction we can now state the motivic version of Thom-Sebastiani theorem:

Theorem 5.29 (Thom-Sebastiani)**.** *We have the following idenities for vanishing cycles*

$$\Phi_{x_1^2+\cdots+x_n^2} = (\Phi_{x^2})^{*k}$$

and therefore

$$\Phi_{x_1^2+\cdots+x_n^2} = (1-\mu_2)^k = \begin{cases} \mathbb{L}^{\frac{k}{2}} & k \text{ is even} \\ \mathbb{L}^{\frac{k-1}{2}}(1-\mu_2) & k \text{ is odd} \end{cases}$$

or if we identify $(1-\mu_2)$ *and* $\mathbb{L}^{\frac{1}{2}}$ *we get*

$$\Phi_{x_1^2+\cdots+x_n^2} = \mathbb{L}^{\frac{k}{2}}$$

always.

5.7 *Homogenous affine morphism–an application of embedded resolutions*

Let $X \cong \mathbb{C}^n \xrightarrow{f_d} \mathbb{C}$ where f_d is any homogeneous morphism of degree d. This means that $f_d^{-1}(0)$ is a cone and that $f_d^{-1}(1) =: X_1$ has a natural μ_d-action via

$$(x_1, \cdots, x_n) \mapsto (\xi_d x_1, \cdots, \xi_d x_n)$$

where $\xi_d \cong e^{\frac{2\pi i}{d}}$.

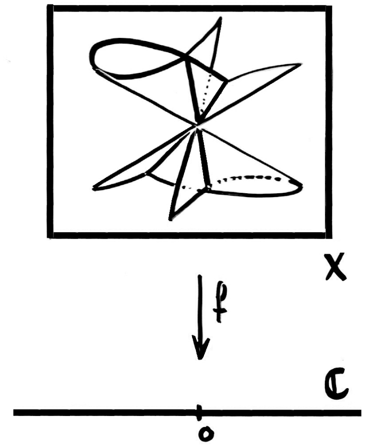


Figure 5.3: Homogenous map of degree d

We define the following integration map $\int_{X_0} : \mathrm{Mot}_{X_0} \to \mathrm{Mot}$ as the push-forward map π_* where $\pi : X_0 \to pt$. Then we have
Proposition 5.30.

$$\int_{X_0} MF_{f_d} = [f_d^{-1}(1), \mu_d].$$

Proof. As first step we blow up $\mathbb{C}^n$ in the origin, and let $\widetilde{X}_0$ be the proper transform of central fiber X_0. Let locus $\{f_d = 0\}$ now intersects the exceptional projective $(n-1)$-plane, which sits as the zero section of the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1) \cong \mathrm{Bl}_0(\mathbb{C}^n)$. We now form the embedded resolution of this singularity as shown in figure 5.4. Note

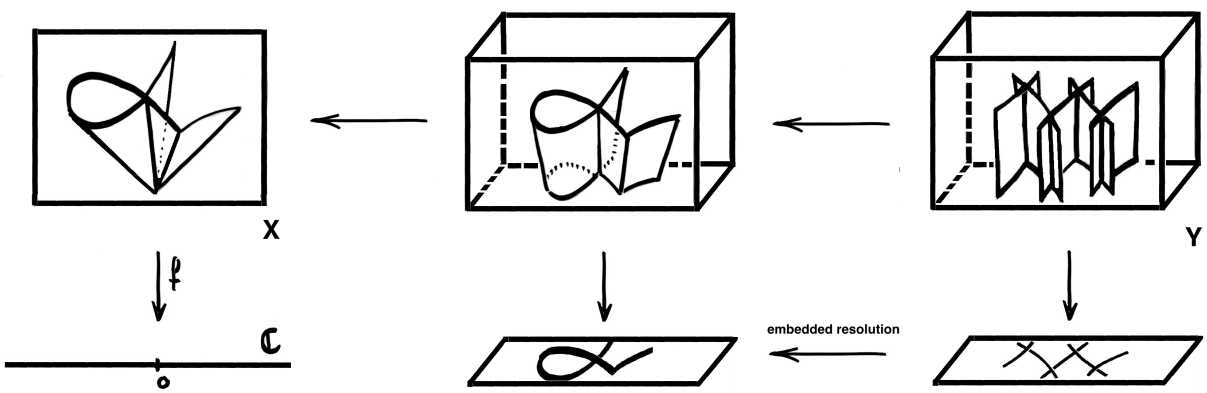


Figure 5.4: Blow up of the cone (the middle fibration) constructs $\widetilde{X}_0$ over the central fiber X_0; the exceptional divisor of X_0 is the base variety of this fibration, $\{f_d = 0\} \subseteq \mathbb{P}^{n-1}$ in the lower middle picture. This is follow by the embedded resolution of singularities, E_0, consisting of normal crossing divisors $F_1, \cdots, F_n$ in lower right hand picture; The pull back (upper right hand picture) creates $E_1, \cdots, E_k$ and $E_0 \subseteq \mathbb{P}^{n-1}$ on the base.

that $E_i \to F_i \cong E_i \cap E_0$ is a $\mathbb{C}$-fibration. And likewise, for any set of indices I, $E_I \to F_{I \cup \{0\}}$ is a $\mathbb{C}$-fibration. Also note that $E_I^0 \to F_{I\cup\{0\}}$ is a $\mathbb{C}^\times$-fibration and hence

$$E_I^0 = (\mathbb{L} - 1)\mathbb{E}^0_{I\cup\{0\}}$$

and after pulling back along f, we get

$$[\widetilde{E_I^0}, \mu_{n_I}] = (\mathbb{L}-1)[\widetilde{E_{I\cup\{0\}}}, \mu_{n_I \cup \{0\}}].$$

Now the integration map send the Milnor fiber to

$$\begin{aligned}\int_{X_0} MF_f &= \sum_{\emptyset\neq J\subseteq\{0,\cdots,k\}} (1-\mathbb{L})^{|J|-1}[\widetilde{E_J^0},\mu_{n_J}] \\ &= \sum_{\emptyset\neq I\subset\{1,\cdots,k\}} (1-\mathbb{L})[\widetilde{E_I^0}] + (1-\mathbb{L})^{|I|}[\widetilde{E_{I\cup\{0\}}^0}] + [\widetilde{E_0^0},\mu_d] \\ &= [\widetilde{E_0^0},\mu_d]\end{aligned}$$

Now we study the motivic class of $\widetilde{E_0^0}$. Note that $\widetilde{E_0^0} \to E_0^0 = \mathbb{P}^{n-1} - \{f_d = 0\}$ is the restriction of the d-fold branched covering of $\mathbb{P}^{n-1}$ branched along $\{f_d = 0\}$.

We can view $f_d|_{\mathbb{P}^{n-1}}$ as a section of $\mathcal{O}(d) \to \mathbb{P}^{n-1}$. So the equation $f_d = s^d$ for $s \in \mathbb{C}$, cut out a subvariety of $\mathcal{O}(1) \to \mathbb{P}^{n-1}$:

$$\iota : \widetilde{E_0^0} = \{f_d = s^d, s \neq 0\} \to \mathcal{O}(1) \setminus \{\text{zero section}\}.$$

Here the ambient space $\mathcal{O}(1) \setminus \{\text{zero section}\}$ has an action of μ_d via $s \mapsto \xi_d s$. We may compose the injection ι with the morphism

$$\mathcal{O}(1) \setminus \{\text{zero section}\} \to \mathcal{O}(-1) \setminus \{\text{zero section}\}$$

via $(x_1, \cdots, x_n, s) \mapsto (x_1, \cdots, x_n, u = \frac{1}{s})$. This idenitifies

$$\{f_d = s^d, s \neq 0\} = \{f_d = u^{-d}, u \neq 0\} \cong \{u^d f_d(x_1, \cdots, x_n) = 1\}$$

but note that f_d is written in homogeneous coordinates, and therefore $f_d(x_1, \cdots, x_n) = f_d(ux_1, \cdots, ux_n)$ hence we conclude that

$$\int_{X_0} MF_{f_d} = [\widetilde{E_0^0}, \mu_d] = [f_d^{-1}(1), \mu_d].$$

$\square$

Mar 26

5.8 *Generalized DT-invariants of Kontsevich and Soibleman*

Let $\mathfrak{M}$ be the moduli stack of objects in $\mathcal{C}oh(X)$ where X is a Calabi-Yau threefold. Recall that given an object E on $\mathcal{C}oh(X)$, $\mathfrak{M}$ is locally written as $\{df_E = 0\}/\operatorname{Aut}(E)$ where $f_E : \operatorname{Ext}^1(E,E) = \operatorname{Def}(E) \to \mathbb{C}$ is the superpotential. Notice that $\operatorname{Lie}(\operatorname{Aut}(E)) \cong \operatorname{Ext}^0(E,E)$.

Theorem 5.31 (Kontsevich-Soiblemann)**.** *There exists a class*

$$[W \to \mathfrak{M}] \in \operatorname{Mot}_{\mathfrak{M}} = K_0^{\hat{\mu}}(\mathsf{St}/\mathfrak{M}) = K_0^{\hat{\mu}}(\mathsf{Var}/\mathfrak{M})[\mathbb{L}^{-1}, (1-\mathbb{L}^n)^{-1}]_{n=1,2,\cdots}$$

such that on each local model $\{df_E = 0\}/\operatorname{Aut}(E)$, *$W$ is given by*

$$\mathbb{L}^{-\frac{1}{2}(\dim \operatorname{Ext}^1(E,E) - \dim \operatorname{Ext}^0(E,E))}\Phi_{f_E}. \tag{5.5}$$

Remark 5.32. Recall that $f : V \to \mathbb{C}$ with $\Phi_f = [V] - MF_f \in \mathrm{Mot}_{\mathrm{crit}(f)}$ and that

$$\mathbb{L}^{\frac{1}{2}} = (1-\mu_2) \in \mathrm{Mot}_{pt}$$
$$e(\mathbb{L}^{\frac{1}{2}}) = e(1-2pt) = -1$$

Therefore the euler characteristic of $W \to \mathfrak{M}$ over a point E is

$$e(W|_E) = (-1)^{\dim \mathrm{Ext}^1(E,E) - \dim \mathrm{Ext}^2(E,E)}(1 - e(MF_{f_E}(0))).$$

Notice how this suggests consistancy with Behrend's weighted euler characteristic, $\nu(E)$:

$$(-1)^{\mathrm{ext}^1(E,E)-\mathrm{ext}^0(E,E)}(1-e(MF_f(0))).$$

As a matter of fact, $W \to \mathfrak{M}$ is called *motivic weight function*.

Ignoring stackiness and automorphisms, the content here is the following: at E we have the local model given by $\mathrm{crit}(f_E)$, and for a neighbor point $E' \in \mathfrak{M}$ we have a different local model described via $\mathrm{crit}(f_{E'}) = \{df_{E'} = 0\} \subset \mathrm{Ext}^1(E', E')$. However the tanget space of $\mathfrak{M}$ is given by $\mathrm{Ext}^1(E,E)$ so we may view $E' \in \mathrm{Ext}^1(E,E)$. If we expand f_E about E', we then get a function beginning with quadratic terms. This means that $E' \in \mathrm{crit}(f_E)$. So after a non-linear change of coordinates on $\mathrm{Ext}^1(E,E)$ we get

$$f_E|_{\text{centered at } E'} = \overbrace{f_{E'}}^{\text{in } \dim \mathrm{Ext}^1(E,E) \text{ variables}} + \underbrace{\text{homogeneous quadratics terms}}_{\dim \mathrm{Ext}^1(E,E) - \dim \mathrm{Ext}^1(E',E') \text{ variables}}.$$

So we apply the motivic Thom-Sebastiani here

$$\Phi_{f_E}|_{\text{centered at} E'} = \Phi_{f_{E'}} * (\mathbb{L}^{\frac{1}{2}})^{\dim \mathrm{Ext}^1(E,E) - \dim \mathrm{Ext}^1(E',E')}.$$

The factor of $\mathbb{L}^{1/2}$ in front of 5.5 compensates for this difference and $W \to \mathfrak{M}$ is independent of choices of local models.

Although there is one important subtly here. We used $\Phi_{x_1^2 + \cdots + x_n^2} = (\mathbb{L}^{\frac{1}{2}})^n$, which we proved in Mot_{pt}. However, In the above argument we need this to be true for families of quadratics over $\mathfrak{M}$. To get an appropriate $\mathbb{L}^{\frac{1}{2}} \in \mathrm{Mot}_{\mathfrak{M}}$ we need a square root $\sqrt{D}$ of

$$D = \mathrm{sdet}(\mathrm{Ext}^\bullet(E,E)) \to \mathfrak{M}.$$

Morally speaking,

$$D|_E = \underbrace{\bigwedge^{top} \mathrm{Ext}^0(E,E) \otimes \bigwedge^{top} \mathrm{Ext}^1(E,E)^\vee}\otimes \underbrace{\bigwedge^{top} \mathrm{Ext}^2(E,E) \otimes \bigwedge^{top} \mathrm{Ext}^3(E,E)^\vee}$$

where the two parts shown are isomorphic. So if $\mathrm{Ext}^i(E,E)$ were locally free then the square root $\sqrt{D}$ existed canonically.

Theorem 5.33 (Kontsevich-Soiblemann)**.** *Let $H(\mathcal{C})$ be the Hall algebra of a Calabi-Yau category $\mathcal{C}$, endowed with orientation data (and consequently $[W \to \mathfrak{M}]$). Then we have a homomorphism of algebras over* $\mathrm{Mot}_{\mathfrak{M}}$

$$\Phi : H(\mathcal{C}) \to \mathrm{Mot}_{\mathfrak{M}}[\![q^{K(\mathcal{C})}]\!]$$

where elements of $\mathrm{Mot}_{\mathfrak{M}}[\![q^{K(\mathscr{C})}]\!]$ *are* $\sum_{\alpha\in K(\mathscr{C})} m_\alpha q^\alpha$ *for* $m_\alpha \in \mathrm{Mot}_{\mathfrak{M}}$ *with*

$$q^\alpha . q^\beta = \mathbb{L}^{\frac{1}{2}\langle\alpha,\beta\rangle} q^{\alpha+\beta}.$$

Here $\langle\alpha,\beta\rangle = \sum_{i=0}^{3}(-1)^i \dim \mathrm{Ext}^i(\alpha,\beta)$ *is a skew-symmetric pairing.*

Specifically,

$$\Phi([X \xrightarrow{\pi} \mathfrak{M}]) = \sum_{\alpha\in K(\mathscr{C})} [X_\alpha \times_{\mathfrak{M}} W] q^\alpha$$

where $X_\alpha = \pi^{-1}(\mathfrak{M}_\alpha)$*. In this manner* Φ *gives the generating function of what we now define as motivic DT-invariants.*

Mar 28

Last time we discussed

1. The existence of motivic weights $W \to \mathfrak{M} \in \mathrm{Mot})\mathfrak{M}$, where $\mathfrak{M}$ is the moduli stack of objects

$$W|_E = \mathbb{L}^{-\frac{1}{2}(\mathrm{Ext}^1(E,E)-\mathrm{Ext}^0(E,E))}\Phi_{f_E}$$

where $f_E : \mathrm{Ext}^1(E,E) \to \mathbb{C}$ is a local model for the moduli space. Here also

$$\mathbb{L}^{\frac{1}{2}} = (1-\mu_2).$$

2. There is a homomorphism

$$\Phi : H(\mathscr{C}) \to \mathrm{Mot}[\![q^{K(\mathscr{C})}]\!].$$

The image ring is the collection of all elements $\sum_{\alpha\in K(\mathscr{C})} m_\alpha q^\alpha$ with the operation

$$q^\alpha . q^\beta = \mathbb{L}^{\frac{1}{2}\langle\alpha,\beta\rangle} q^{\alpha+\beta}$$

via $\langle\alpha,\beta\rangle = \chi(\alpha,\beta)$. The integration map is defined such that

$$\Phi([X \to \mathfrak{M}]) = \sum_{\alpha\in K(\mathscr{C})} X_\alpha \times_{\mathfrak{M}} W q^\alpha$$

where $X_\alpha = \pi^{-1}(\mathfrak{M}(\alpha))$.

Example 5.34. Let $I = [I_\bullet(X,\bullet) \to \mathfrak{M}]$ via $[\mathscr{O}_X \twoheadrightarrow \mathscr{O}_C] \mapsto \mathscr{O}_C$ where

$$I_\bullet(X,\bullet) = \bigsqcup_{n,\beta} I_n(X,\beta).$$

We have

$$\Phi(I) = \sum_{n,\beta} [I_n(X,\beta) \times_{\mathfrak{M}} W] q^n v^\beta.$$

This defines for us the motivic Donaldson-Thomas invariants[3]

$$Z_X^{DT,\mathrm{mot}} = \Phi(I).$$

[3] Strictly speaking we want the weights associated to $I_n(X,\beta)$, not $\mathfrak{M}$.

If we apply the Poincare polynomial to the image of integration, we get the refined invariants

$$Z_X^{DT,\mathrm{ref}} = P_t(\Phi(I)).$$

To get the numerical invariants we further apply the euler characteristic:

$$Z_X^{DT,\mathrm{num}} = e(P_t(\Phi(I)))$$

when this is possible! This is a composition through the following rings

$$\mathrm{Mot}\,[\![q^{K(\mathscr{C})}]\!] \xrightarrow{P_t} \mathbb{Z}[t,t^{-1},(1-t^m)^{-1}][\![q^{K(\mathscr{C})}]\!] \xrightarrow{e} \mathbb{Z}[\![q^{K(\mathscr{C})}]\!].$$

5.9 *Applications*

5.9.1 *Finishing Bridgeland's proof*

Recall that we derived identity 5.4 in the Hall algebra of $\mathscr{C} = \mathcal{C}oh_{\leq 1}(X)$ for a Calabi-Yau threefold X:

$$I * 1_{\mathfrak{P}} * 1_{\mathfrak{Q}} = I_0 * 1_{\mathfrak{P}} * PT * 1_{\mathfrak{Q}}. \tag{5.6}$$

Now we apply our integration map. We get

$$Z_0^{DT,\mathrm{mot}}.\Phi(1_{\mathfrak{P}}).Z^{PT,\mathrm{mot}}.\Phi(1_{\mathfrak{Q}}) = Z^{DT,\mathrm{mot}}\Phi(1_{\mathfrak{P}})\Phi(1_{\mathfrak{Q}}) \in \mathrm{Mot}\,[\![q^{K(\mathscr{C})}]\!].$$

But because $\langle\bullet,\bullet\rangle$ is zero on $K(\mathscr{C})$ the quantum torus is commutative in this case (note that through Serre duality, $\langle\bullet,\bullet\rangle$ works like Poincare pairing). This completes the proof of

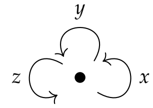
$$Z^{PT,\mathrm{mot}} = \frac{Z^{DT,\mathrm{mot}}}{Z_0^{DT,\mathrm{mot}}}.$$

This in particular holds for refined and numerical invariants.

5.9.2 *Degree zero motivic DT invariants*

Let $\mathscr{C} = \mathcal{C}oh_{\leq 0}(\mathbb{C}^3)$. Understanding this allows us to understand $\mathcal{C}oh_{\leq 0}(X)$ for any Calabi-Yau threefold. Firstly Note that $\mathfrak{M}_n$ is at the same time moduli stack of the following categories

- zero dimensional coherent sheave fo lengths n on $\mathbb{C}^3$;
- modules W over $\mathbb{C}[x,y,z]$ with $\dim_{\mathbb{C}} W = n$;
- triples (x,y,z) of endomorphisms of $\mathbb{C}^n$ as vector space such that $[x,y] = [x,z] = [y,z] = 0$;
- representations of the following quiver with

  relations $[x,y] = [x,z] = [y,z] = 0$;

- flat rank n vector bundles on T^3.

We will work with the most concrete one:

$$\mathfrak{M}_n = \{(x,y,z) \in (M_{n\times n}(\mathbb{C}))^3 : [x,y] = [x,z] = [y,z] = 0\} / \operatorname{GL}_n .$$

Let $f_n : M^3_{n\times n} \to \mathbb{C}$ be given via $(x,y,z) \mapsto \operatorname{tr}(x[y,z])$. Then

$$\{df_n = 0\} = \{[x,y] = [x,z] = [y,z] = 0\}$$

and $\mathfrak{M}_n = \{df_n = 0\} / \operatorname{GL}_n$ and in fact this is the local model of $\mathfrak{M}_n$ at the origin, i.e. at $E = \mathcal{O}_0^{\oplus n}$. Here $\mathcal{O}_0$ is the skyscraper sheave at the origin and one can check that

$$f_n : \operatorname{Ext}^1(\mathcal{O}_0^{\oplus n}, \mathcal{O}_0^{\oplus n}) \cong M^3_{n\times n} \to \mathbb{C}$$

is the same as the above expression. In fact this is a global description of $\mathfrak{M}_n$ in this case as well.

Our motivic invariant is

$$\begin{aligned} Z^{\mathfrak{M}} &= \Phi(1_{\mathfrak{M}}) = \sum_n [W_n] q^n \\ &= \sum_{n=0}^{\infty} [\mathbb{L}^{-\frac{1}{2}(3n^2-n^2)} \Phi_{f_n}] / [\mathbb{G}L_n] q^n \end{aligned}$$

because $\dim \operatorname{Ext}^1 = 3n^3$ and $\dim \operatorname{Ext}^0 = n^2$. We now need to find the vanishing cycle of f_n. Note that by the expression of $f_n : (\mathbb{C}^{n^2})^3 \to \mathbb{C}$, it is a homogeneous polynomial of degree 3. Hence,

$$\begin{aligned} \Phi_{f_n} &= [f_n^{-1}(0)] - [f_n^{-1}(1), \mu_3] \\ &= \mathbb{L}^{-n^2}([f_n^{-1}(0)] - [f_n^{-1}(1)]) \end{aligned}$$

Thus

$$\begin{aligned} f_n^{-1}(0) - f_n^{-1}(1) &= f_n^{-1}(0) - \frac{1}{\mathbb{L}-1} f_n^{-1}(\mathbb{C}^\times) \\ &= \frac{1}{\mathbb{L}-1}((\mathbb{L}-1) f_n^{-1}(0) - (\mathbb{L}^{3n^2} - f_n^{-1}(0))) \\ &= \frac{1}{\mathbb{L}-1}(\mathbb{L} f_n^{-1}(0) - \mathbb{L}^{3n^2}) \end{aligned}$$

Now we find the motivic class of $f_n^{-1}(0)$ by stratifying the zero locus of $\operatorname{tr}(x[y,z])$ in two strata: one, if $[y,z] = 0$, in which case x can be anything, and the other one, where $[y,z] \neq 0$ and x is orthogonal to it. So we have

$$\begin{aligned} f_n^{-1}(0) &= \mathbb{L}^{n^2} C_n + \mathbb{L}^{n^2-1} \underbrace{(\mathbb{L}^{2n^2} - C_n)}_{[y,z]\neq 0} \\ &= \mathbb{L}^{n^2-1}(\mathbb{L} C_n + \mathbb{L}^{2n^2} - C_n) \end{aligned}$$

where C_n is the motive of the commuting endomorphisms. We have

$$\mathbb{L}^{-n^2}(f_n^{-1}(0) - f_n^{-1}(1)) = C_n.$$

So we get

$$Z^{\mathfrak{M}}(q) = \sum_{n=0}^{\infty} \frac{[C_n]}{[\mathrm{GL}_n]} q^n.$$

This series was computed by Feit-Fine in 1940! They were interested in the question of finding the probability that two $n \times n$ matrices over $\mathbb{F}_p$ commute? Their proof amounts to stratifying the variety C_n by strata isomorphic to $\mathbb{A}^{k_i}_{\mathbb{F}_p}$. For them $\#[\mathbb{A}_{\mathbb{F}_p}] = p$, hence if we substitute $\mathbb{L}$ for p in their formula we get the beautiful formula

$$\sum_{n=0}^{\infty} \frac{[C_n]}{[\mathrm{GL}_n]} q^n = \prod_{m=1}^{\infty} \prod_{j=0}^{\infty} (1 - \mathbb{L}^{1-j} t^m)^{-1} \in \mathrm{Mot}[\![t]\!] \subset K_0(\mathsf{Var})[\![\mathbb{L}^{-1}]\!].$$

Mar 31

Last time we considered $\mathscr{C} = \mathcal{C}oh_{\leq 0}(\mathbb{C}^3)$. The motivic weight $W_n \to \mathfrak{M}_n$ here is $\mathbb{L}^{-n^2}\Phi_{f_n} = W_n$. We computed, $Z^{\mathfrak{M}}$, the motivic DT-invariants of $\mathfrak{M}$ and quoted a result of Feit and Fine

$$Z^{\mathfrak{M}} = \prod_{m=1}^{\infty} \prod_{j=0}^{\infty} (1 - \mathbb{L}^{1-j} t^m)^{-1} \in \mathbb{Z}[\mathbb{L}][\![\mathbb{L}^{-1}, q]\!].$$

In following sessions we give a proof of this using power structures.

Up to second order terms we have

$$\begin{aligned} Z^{\mathfrak{M}} &= \prod_{j=0}^{\infty} (1 + \mathbb{L}^{1-j} q + o(q^2)) \\ &= 1 + (\sum_{j=0}^{\infty} \mathbb{L}^{1-j}) q + o(q^2) \end{aligned}$$

So the coefficient of q is $\sum_{j=0}^{\infty} \mathbb{L}^{1-j} = \frac{\mathbb{L}^2}{\mathbb{L}-1}$ which is, as expected, $[C_1]/[\mathrm{Gl}_1]$. Note that $\mathfrak{M}$ is a stack (not a scheme) and therefore the specialization $\mathbb{L} \to 1$ is not well-defined.

Traditional DT invariants of $\mathbb{C}^3$ come from $\mathrm{Hilb}^n(\mathbb{C}^3) = I_0(\mathbb{C}^3, 0)$:

$$N_n^{DT} = e_{vir}(I_n(\mathbb{C}^3, 0)).$$

As per previous notation, let

$$I_0 = [\sqcup_n I_n(\mathbb{C}^3, 0) \to \mathfrak{M}_n] \in H(\mathscr{C})$$

where the structure morphism over $\mathfrak{M}_n$, maps $\mathscr{O}_{\mathbb{C}^3} \twoheadrightarrow \mathscr{O}_Z$ to $\mathscr{O}_Z \in \mathfrak{M}_n$. Also let

$$I^{\mathscr{O}}_{\mathfrak{M}} = [\mathfrak{M}(\mathscr{O}) \to \mathfrak{M}]$$

via $(\mathcal{O} \to \mathcal{F}) \mapsto \mathcal{F}$. Recall that the content of Hall algebra identity $I^{\mathcal{O}}_{\mathfrak{M}} = I_0 * I_{\mathfrak{M}}$ is that every morphism factors as a surjection followed by an injection:

$$\begin{tikzcd} & & \mathcal{O} \arrow[dl] \arrow[d] & & \\ 0 \arrow[r] & \mathcal{O}_Z \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{Q} \arrow[r] & 0. \end{tikzcd}$$

Note that $\mathfrak{M}_n(\mathcal{O}) \to \mathfrak{M}_n$ is a fibration with fiber $\mathbb{C}^n$ and hence $\mathfrak{M}_n(\mathcal{O}) \times_{\mathfrak{M}_n} W$ is a vector bundle over W_n, therefore:

$$[\mathfrak{M}_n(\mathcal{O}) \times_{\mathfrak{M}_n} W] = \mathbb{L}^n[W_n].$$

Consequently $\Phi(1^{\mathcal{O}}_{\mathfrak{M}}) = Z^m(\mathbb{L}q)$ and

$$Z^m(\mathbb{L}q) = \Phi(I_0)Z^m(q) \in \mathrm{Mot}[\![q]\!]$$

where $\mathrm{Mot}[\![q]\!]$ is commutative. So we have

$$\Phi(I_0) = \frac{Z^m(\mathbb{L}q)}{Z^m(q)}.$$

The formula

$$\Phi(I_0) = \sum_{n=0}^{\infty} [I_n(\mathbb{C}^3, 0) \times_{M_n} W_n] q^n$$

is close to

$$Z_0^{DT,\mathrm{mot}}(\mathbb{C}^3) = \sum_{n=0}^{\infty} [W_n \to I_n(\mathbb{C}^3, 0)] q^n$$

where W_n is the weight function coming from I_n.

So we look at $\mathrm{Ext}^\bullet(\mathcal{O} \to \mathcal{O}_Z, \mathcal{O} \to \mathcal{O}_Z)$ versus $\mathrm{Ext}^\bullet(\mathcal{O}_Z, \mathcal{O}_Z)$. These differ by $\mathrm{Ext}^0(\mathcal{O}_Z, \mathcal{O}_Z) = \mathbb{C}^n$ only by automorphisms. Hence we have

$$[W_n \to I_n(\mathbb{C}^3, 0)] = \mathbb{L}^{-\frac{n}{2}}[I_n(\mathbb{C}^3, 0) \times_{\mathfrak{M}_n} W_n]$$

and therefore

$$Z_0^{DT,\mathrm{mot}}(\mathbb{C}^3) = \Phi(\mathbb{L}^{-\frac{1}{2}}q) = \frac{Z^m(\mathbb{L}^{\frac{1}{2}}q)}{Z^m(\mathbb{L}^{\frac{1}{2}})}.$$

So we apply

$$Z^m(q) = \prod_{m=1}^{\infty}\prod_{j=0}^{\infty}(1 - \mathbb{L}^{1-j}q^m)^{-1}$$

to it and we get

$$\begin{aligned} Z_0^{DT,\mathrm{mot}}(\mathbb{C}^3) &= \prod_{m=1}^{\infty}\prod_{j=0}^{\infty}\frac{(1 - \mathbb{L}^{1-j+m/2}q^m)^{-1}}{(1 - \mathbb{L}^{1-j-m/2}q^m)^{-1}} \\ &= \prod_{m=1}^{\infty}\prod_{j=0}^{m-1}(1 - \mathbb{L}^{1-j+m/2}q^m)^{-1} \end{aligned}$$

and remember that this can be used to find specializations to refined and numerical invariants:

$$
\begin{aligned}
Z_0^{DT,\mathrm{ref}}(\mathbb{C}^3) &= \prod_{m=1}^{\infty}\prod_{j=0}^{m-1}(1-t^{2-2j+m}q^m)^{-1} \\
Z_0^{DT}(\mathbb{C}^3) &= \prod_{m=1}^{\infty}\prod_{j=0}^{m-1}(1-(-q)^m)^{-1} \\
&= \prod_{m=1}^{\infty}(1-(-q)^m)^{-m} = M(-q).
\end{aligned}
$$

and notice we have not used the torus action at all. So this can be regarded as an independent proof for computation of this numerical invariant without using the box counting argument.

We can use this method to compute invariants for objects in the whole category, but before doing that we present another example first.

Example 5.35. Let C be a curve of genus g in a Calabi-Yau threefold X. For example we can take a curve C and consider the total space of a generic rank two vector bundle $N \to X$ such that $\wedge^2 N \cong K_C$. The PT-moduli space is

$$
\begin{aligned}
PT_n(X,[C]) &= \left\{ \text{non-zero } \mathcal{O}_X \to s_* L, \begin{array}{c} L \to C \text{ is line bundle} \\ \text{and } n = \deg L + 1 - g \end{array} \right\} \\
&= \mathrm{Sym}^{n+g-1}(C).
\end{aligned}
$$

hence smooth of dimension $n+g-1$. The moduli space is unobstructed, and the local Chern-Simon potential

$$
f : \mathrm{Ext}^1(\mathcal{O}_{\to} s_* L, \mathcal{O}_x \to s_* L) \to \mathbb{C}
$$

is zero. The motivic weight is $\mathbb{L}^{-1/2(n+g-1)}$. So

$$
\begin{aligned}
Z_{[C]}^{PT,\mathrm{mot}} &= \sum_{n=1-g}^{\infty} \mathbb{L}^{-1/2(n+g-1)}[\mathrm{Sym}^{n+g-1}(C)]q^n \\
Z_{[C]}^{PT,\mathrm{ref}}(X) &= \sum_{n=1-g}^{\infty} t^{-(n+g-1)} P_t(\mathrm{Sym}^{n+g-1}(C)) q^{n+g-1} q^{1-g} \\
& q^{1-g}\sum_{k=0}^{\infty} P_t(\mathrm{Sym}^k(C))(t^{-1}q)^k
\end{aligned}
$$

Here the Poincare polynomial is the power series of Betti numbers $P_t(s) = \sum_\ell b_\ell s^\ell$. Now we use this theorem of MacDonald:

Theorem 5.36 (MacDonald)**.**

$$
\sum_{k=0}^{\infty} P_t(\mathrm{Sym}^k S) z^k = \frac{\prod_{odd\ \ell}(1+t^\ell z)^{b_\ell}}{\prod_{even\ \ell}(1-t^\ell z)^{b_\ell}}
$$

So
$$\sum_{k=0}^{\infty} P_t(\mathrm{Sym}^k(C))z^k = \frac{(1+tz)^{2g}}{(1-z)(1-t^2z)}.$$
and we get the refined invariants

$$\begin{aligned} Z^{PT,\mathrm{ref}}_{[C]}(X) &= \frac{(1-q)^{2g}q^{-g}}{q^{-1}(1-tq)(1-t^{-1}q)} \\ &= \frac{(q^{-1}+q+2)^g}{(q+q^{-1})-(t+t^{-1})} \end{aligned}$$

and notice that this is symmetric under $q \leftrightarrow q^{-1}$ and we can do the MNOP substitution in it. Also by letting $t \to -1$ we get the numerical PT-invariants to be $(q^{1/2}+q^{-1/2})^{2g-2}$.

April 1

5.10 *Power structures and their applications*

Last time we computed $Z_0^{DT,\mathrm{mot}}(\mathbb{C}^3)$, i.e. virtual motives of $\mathrm{Hilb}^n(\mathbb{C}^3)$. We now want to use a *power structure* on $K_0(\mathsf{Var}/\mathbb{C})$ to compute $Z_0^{DT,\mathrm{mot}}(X)$ where X is any Calabi-Yau threefold.

Definition 5.37. A power structure on a commutative unital ring R is a map
$$R \times (1_t R[\![t]\!]) \to 1 + tR[\![t]\!]$$
given by $(m, A(t)) \mapsto A(t)^m$, satisfying
1. $A(t)^0 = 1$,
2. $A(t)^1 = A(t)$,
3. $(A(t)B(t))^m = A(t)^m B(t)^m$,
4. $(A(t))^{m+n} = A(t)^m A(t)^n$,
5. $(A(t))^{mn} = ((A(t))^m)^n$,
6. $(1+t)^m = 1 + mt + o(t^2)$,
7. $A(t^k)^m = (A(s)^m)|_{s=t^k}$ where k is a natural number.

More or less this is equivalent to a pre-lambda ring structure on R.

Theorem 5.38 (Gusein-Zade, Melle-Hernandez)**.** *There exists a power structure on $K_0(\mathsf{Var}/\mathbb{C})$ uniquely determined by*
$$(1-t)^{-[X]} = \sum_{n=0}^{\infty} [\mathrm{Sym}^n X] t^n.$$

So note that number (7) in the definition implies that
$$(1-t^2)^{-[X]} = \sum [\mathrm{Sym}^n X] t^{2n}$$
but it is not true that
$$(1+t)^{-[X]} = \sum_{n=0}^{\infty} (-1)^n [\mathrm{Sym}^n(X)] t^n.$$

More generally, if $A(t) = \sum A_i t^i$ where A_i's are varieties and X is a variety, then

$$(A(t))^X = \sum_{k=0}^{\infty} B_k t^k$$

where B_k is the space of configurations of A_i-valued points on X of total charge k, i.e. set of pairs (S, φ) where $S \subset X$ is a finite subset and $\varphi : S \to \sqcup_i A_i$, such that $k = \sum_{x \in S} i(\varphi(x))$. We can write this as

$$B_k = \bigsqcup_{\alpha \vdash k} \left(\prod_{i=1}^{\infty} X^{b_i(\alpha)} \smallsetminus \Delta \right) \times_{S_\alpha} \left(\prod_{i=1}^{\infty} A_i^{b_i(\alpha)} \right)$$

where $b_i(\alpha)$ is the number of i's in the partition α and $S_\alpha = \prod_{i=1}^{\infty} S_{b_i(\alpha)}$.

Lemma 5.39 (Totaro)**.** *In $K_0(\mathsf{Var}/\mathbb{C})$ we have $[\mathrm{Sym}^n \mathbb{C}^k] = [\mathbb{C}^{nk}]$. So in terms of power structures this means that*

$$(1 - t^m)^{-\mathbb{L}^k} = (1 - \mathbb{L}^k t^m)^{-1}.$$

Aside: Under the Poincare polynomial specialization $K_0(\mathsf{Var}/\mathbb{C}) \to \mathbb{Z}[s]$ we have $\mathbb{L} \mapsto s^2$. Henc the power structure specializes to a power stsructure on $\mathbb{Z}[s]$ satisfying

$$(1 - t)^{-(-s)^k} = (1 - (-s)^k t)^{-1}.$$

This helps us to recover the formula of MacDonald as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} [\mathrm{Sym}^n X] t^n &= (1-t)^{-[X]} \\
\sum_{n=0}^{\infty} P_s(\mathrm{Sym}^n X) t^n &= (1-t)^{-\sum b_k s^k} \\
&= \prod_k (1-t)^{-b_k (-1)^k (-s)^k} \\
&= \prod_k (1 - (-s)^k t)^{-b_k (-1)^k} \\
&= \frac{\prod_{k \text{ is odd}} (1 + s^k t)^{b_k}}{\prod_{k \text{ is even}} (1 - s^k t)^{b_k}} \\
&= (1-t)^{-e(x)} \text{ if we take euler characteristics by } s = -1.
\end{aligned}$$

April 4

Last time we introduced power structures:

$$R \times (1 + tR[\![t]\!]) \to 1 + tR[\![t]\!].$$

So when $R = K_0(\mathsf{Var}/\mathbb{C})$, there exists a power structure such that if $A(t) = \sum_{i=0}^{\infty} A_i t^i$, with A_i being varieties, then for a variety X, we have

$$A(t)^X = \sum B_i t^i$$

where informally speaking B_k is the space of systems of particles of total charge k where the internal state space of a charge i particular is A_i, and $\sum i_\ell = k$. For example we saw that

$$(1+t+t^2+\cdot)^{[X]} = (1-t)^{-[X]} = \sum_k \operatorname{Sym}^k X t^k$$

and we found

$$(1-t)^{-\mathbb{L}^k} = (1-\mathbb{L}^k t)^{-1}.$$

What we want is not just a power structure on the Grothendieck group of varieties, but more generally that of stacks where motives of general linear groups are inverted.

Lemma 5.40. *The power structure on $K_0(\mathsf{Var}/\mathbb{C})$ extends to $K_0(\mathsf{Var}/\mathbb{C})[\![L^{-1}]\!]$, such that $(1-t)^{-\mathbb{L}^{-k}} = (1-\mathbb{L}^{-k}t)^{-1}$ and such that if $A(t) = \sum_{i=0}^\infty A_i t^i$ where A_i are now stacks, and X is still a variety, then the above characterization still holds; namely, $A(t)^X = \sum B_i t^i$, where B_i is the configuration of A_i-valued point son X as before.*

Note that $K_0(\mathsf{Var}/\mathbb{C})[\![L^{-1}]\!]$ contains $K_0(\mathsf{St}/\mathbb{C}) = K_0(\mathsf{Var}/\mathbb{C})[L^{-1},(1-\mathbb{L}^n)^{-1}]$.

5.10.1 *Application 1 - Feit-Fine formula*

As an application we will now give a simple proof of Feit-Fine formula: recall that in our notation C_n is the scheme of commuting matrices $\{[A,B]=0\} \subset \mathbb{C}^(2n^2)$, and $C_n/\operatorname{GL}_n \cong M_n(\mathbb{C}^2)$ is the stack of sheaves of dimension zero and length n on $\mathbb{C}^2$. We define

$$C(t) = \sum_{n=0}^\infty \frac{[C_n]}{[\operatorname{GL}_n]} t^n = \sum_{n=0} \infty \left[\frac{C_n}{\operatorname{GL}_n}\right] t^n.$$

This means that

$$\begin{aligned} C(t) &= \sum_{n=0}^\infty [M_n(\mathbb{C}^2)] t^n \\ &= \left(\sum_{n=0}^\infty [M_n^{(0,0)}(\mathbb{C}^2)] t^n\right)^{[\mathbb{C}^2]} \end{aligned}$$

Here the notation $M_n^{(x_0,y_0)}(\mathbb{C}^2)$ is for the space of sheaves supported at the point (x_0,y_0). Note that this means multiplication by $(x-x_0)^N$ and $(y-y_0)^N$ is zero for large enough N. Hence $A - x_0\,\mathrm{id}$ and $B - y_0\,\mathrm{id}$ are nilpotent, and eigenvalues of A are all x_0 and eigenvalues of B are all y_0.

So we have

$$\left(\sum_{n=0}^\infty [M_n^{(0,*\neq 0)}(\mathbb{C}^2)] t^n\right) = \left(\sum_{n=0}^\infty [M_n^{(0,0)}(\mathbb{C}^2)] t^n\right)^{[\mathbb{C}\setminus\{0\}]}$$

and hence

$$C(t) = \left(\sum_{n=0}^{\infty} [M_n^{(0,*\neq 0)}(\mathbb{C}^2) t^n\right)^{\frac{\mathbb{L}^2}{\mathbb{L}-1}}.$$

Note that

$$\begin{aligned} M_n^{(0,*\neq 0)}(\mathbb{C}^2) &= \{[A,B]=0, A \text{ is nilpotent and } B \text{ is invertible }\}/\operatorname{GL}_n \\ &= \bigsqcup_{\lambda \vdash n} \{J_\lambda = BJ_\lambda B^{-1}\}/\operatorname{Stab}_{J_\lambda}(\operatorname{GL}_n) \end{aligned}$$

and here $[M_n^{(0,*\neq 0)}(\mathbb{C}^2)]$ is a finite number of copies of motives of points, and the cardinality of it is $p(n)$, the number of partitions of n. So now we finish with

$$\begin{aligned} C(t) &= \left(\sum_{n=0}^{\infty} p(n) t^n\right)^{\frac{\mathbb{L}^2}{\mathbb{L}-1}} = \left(\prod_{m=1}^{\infty}(1-t^m)^{-1}\right)^{\sum_{k=1}^{\infty} \mathbb{L}^{2-k}} \\ &= \prod_{k=1}^{\infty}\prod_{m=1}^{\infty}(1-t^m)^{-\mathbb{L}^{2-k}} = \prod_{k=1}^{\infty}\prod_{m=1}^{\infty}(1-\mathbb{L}^{2-k}t^m)^{-1}. \end{aligned}$$

5.10.2 *Application 2 - motivic invariants of Hilbert schemes of points*

As we promised, we can now compute the motives of Hilbert schemes of points on smooth X of dimension d. We start with

$$\begin{aligned} \sum_{n=0}^{\infty} [\operatorname{Hilb}^n(X)] t^n &= \left(\sum_{n=0}^{\infty} [\operatorname{Hilb}^n_{(0)}(\mathbb{C}^d)] t^n\right)^{[X]} \\ &= \left(\sum_{n=0}^{\infty} [\operatorname{Hilb}^n(\mathbb{C}^d)] t^n\right)^{[X].\mathbb{L}^{-d}} \end{aligned}$$

For a Calabi-Yau threefold X we now have,

$$Z_0^{DT,\text{mot}}(X) = \sum_{n=0}^{\infty} [W \to I_n(??)] t^n$$

but W is also local in nature so

$$\begin{aligned} Z_0^{DT,\text{mot}}(X) &= \left(\sum_{n=0}^{\infty} [W \to I_n(\mathbb{C}^3, 0)] t^n\right)^{[X]\mathbb{L}^{-3}} \\ &= \left(\prod_{m=1}^{\infty}\prod_{j=0}^{m-1}(1-\mathbb{L}^{m/2+1-k})\right)^{[X]\mathbb{L}^{-3}} \end{aligned}$$

So we get that

$$Z_0^{DT,\text{mot}}(X) = \prod_{m=1}^{\infty}\prod_{j=0}^{m-1}(1-\mathbb{L}^{m/2+1-k})^{[X]}.$$

Now for specializations we have the Poincare polynomial

$$P_s[X] = \sum_{d=0}^{6} b_d s^d = \sum_{d=0}^{6} b_d (-1)^d (-s)^d$$

so as we substitute $\mathbb{L}^{1/2}$ with s we get the refined invariants

$$\begin{aligned} Z_0^{DT,\mathrm{ref}}(X) &= \prod_{d=0}^{6} \prod_{m=1}^{\infty} \prod_{j=1}^{m-1} \left(1 - (-s)^{m-4-2j}(-t)^m\right)^{(-1)^d b_d (-s)^d} \\ &= \prod_{d=0}^{6} M_{d-3}(-s,-t)^{(-1)^d b_d} \end{aligned}$$

where

$$M_\delta(s,t) = \prod_{m=1}^{\infty} \prod_{j=1}^{m} \left(1 - s^{m-2j+1+\delta} t^m\right)^{-1}.$$

This is the analogue of MacDonald and Göttche's formula for Poincare polynomial of Hilbert schemes of n points over curves and surfaces. It is not clear if this picture can go on in dimension four, as those schemes do not have a virtual fundamental class the way we need.

April 7

5.10.3 *Case of local curves*

Now we want to consider local curves $X = \mathrm{tot}(\mathcal{L}_1 \oplus L_2)$ where $L_1 \oplus L_2 \to C$ is a rank two bundle over a smooth curve of genus g and $L_1 \otimes L_2 \cong K_C$. Solving the curve-counting invariants of local curves is an important but hard problem (but doable)! We will find $Z_{d[C]}^{PT/DT/GW}(X)$. We will specifically compute $Z_{d[C]}^{DT}(\mathrm{tor}(K^{1/2} \oplus K^{1/2})$ for these local curves.

Park and Ionel, show that on a generic almost complex structure on a Calabi-Yau threefold, the pseudo-holomorphic curves are "almost" isolated. And using this they show that the general Gupakumar-Vafa conjecture reduces to local curves with $L_1 = L_2 = K_C^{1/2}$. One can think of such local geometries, as the *idealized* neighborhood of a curve in a general Calabi-Yau threefold.

Using power structures we obtain an easy proof[4] of the formula for

$$Z_d^{DT}(X) = \sum_n e(I_n(X,d][C]), \nu_n) q^n$$

where $\nu_n : I_n(X, d[C]) \to \mathfrak{Z}$ is the Behrend function and recall that Behrend shows that ν_n only depends on $I_n(X, d[C])$ as a scheme. X has a $T = \mathbb{C}^\times \times \mathbb{C}^\times$-action. So T acts on $I_n(X, C)$ and given that ν_n only depends on the scheme structures, it has to be constant on orbits of T. So we may stratify $I_n(X, d[C])$ by dimension of orbits, and all the

[4] By ignoring some Behrend function issues.

ones that are not fixed-points have zero weighted euler characteristics, so we have

$$Z_d^{DT}(X) = \sum_n e(I_n(X, d[C])^T, \nu_n) q^n.$$

What do T-invariant subschemes of X look like? It is a d-fold thickening of the curve supported on the zero section and it has embedded points and its generic slice with a fiber is a T-invariant of $\mathbb{C}^2$, hence corresponding to a monomial ideal, hence corresponding to a partition $\alpha \vdash d$ of integer d. We use the notation $(i,j) \in \alpha$ for $(i,j) \in \mathbb{Z}_{\geq 0}^{\oplus 2}$ if and only if $x^i y^j \in \mathcal{O}_Z = \mathbb{C}[x,y]/I_Z$ where I_Z is the associated monomial ideal.

Pure subschemes which are T-invariant are in bijection with $\alpha \vdash d$. Note that we may write the scheme structure of X as

$$X = \mathrm{Spec}_C(\mathrm{Sym}^\bullet(L_1^\vee \oplus L_2^\vee)).$$

So for a pure, T-invariant subschemes C_α corresponding to some partition $\alpha \vdash d$, the structure sheaf is

$$\mathcal{O}_{C_\alpha} = \bigoplus_{(i,j)\in\alpha} L_1^{-i} \otimes L_2^{-j}.$$

A general T-invariant subscheme Z will have an ideal sheaf, I_Z, fitting in a short exact sequence

$$0 \to I_Z \to I_{C_\alpha} \to P \to 0$$

where P is supported at points, and $n = \chi(\mathcal{O}_{C_\alpha}) + (length)(P)$. So thus get

$$\begin{aligned} Z_d^{DT}(X) &= \sum_n e(I_n(X, d[C])^T, \nu_n) q^n \\ &= \sum_{\alpha \vdash d} q^{\chi(\mathcal{O}_{C_\alpha})} \sum_{k=0}^{\infty} e(I_k(X, C_\alpha)^T, \nu_n) q^k \end{aligned}$$

here C_α is the underlying pure subschemes, and k is the length of the embedded points. The claim is that $\nu_n = (-1)^n$ on $I_k(X, C_\alpha)^T$.

If we let $Q = -q$ then we get

$$Z_d^{DT}(X) = \sum_{\alpha \vdash d} Q^{\chi(\mathcal{O}_{C_\alpha})} \sum_{k=0}^{\infty} e(I_k(X, C_\alpha)^T) Q^k.$$

The motive of $I_k(X, C_\alpha)^T$ satisfies a power structure relation:

$$\sum_{k=0}^{\infty} [I_k(X, C_\alpha)^T] t^k = \left(\sum_{k=0}^{\infty} [I_k(\mathbb{C}^3, C_\alpha)_0^T] t^k \right)^{[C]}.$$

Here the subscript 0, encodes the requirement for embedded points to be only supported at the origin. We derive that

$$Z_d^{DT}(X) = \sum_{\alpha \vdash d} Q^{\chi(\mathcal{O}_{C_\alpha})} \left(\sum_{k=0}^{\infty} e(I_k(\mathbb{C}^3, C_\alpha)_0^T) Q^k \right)^{2-2g}.$$

We compute the inside sum now

$$\sum_{k=0}^{\infty} e(I_k(\mathbb{C}^3, C_\alpha)_0^T)Q^k = \sum_{k=0}^{\infty} e(I_k(\mathbb{C}^3, C_\alpha)^{\mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times})Q^k$$
$$= \sum_{k=0}^{\infty} \# \begin{pmatrix} \text{ways of adding } k \text{ boxes} \\ \text{to configuration of boxes} \\ \text{along } x \text{ axis of profile } \alpha \end{pmatrix} Q^k$$

But not that this is the familiar topological vertex $V_{\emptyset\emptyset\alpha}(Q)$, and by the formula of Vafa-Okounkov-Reshetikhin it is

$$V_{\emptyset,\emptyset,\alpha}(Q) = M(Q) \prod_{(i,j)\in\alpha} \frac{1}{1 - Q^{h_\alpha(i,j)}}$$

where $h_\alpha(i,j)$ is the hook length of partition α at node (i,j). We therefore have

$$Z_d^{DT}(X) = \sum_{\alpha\vdash d} Q^{\chi(\mathcal{O}_{C_\alpha})}(V_{\emptyset\emptyset\alpha}(Q))^{2-2g}$$
$$Z_0^{DT}(X) = M(-q)^{e(X)} = M(Q)^{2-2g}$$
$$\frac{Z_d^{DT}(X)}{Z_0^{DT}(X)} = Z_d^{PT} = \sum_{\alpha\vdash d} Q^{\chi(\mathcal{O}_{C_\alpha})} \left(\prod_{(i,j)\in\alpha} \frac{1}{1 - Q^{h_\alpha(i,j)}} \right)^{2-2g}$$

We need the Euler characteristic as well given that $L_1 \cong L_2 \cong K_C^{1/2}$,

$$\chi(\mathcal{O}_{C_\alpha}) = \chi\Big(\bigoplus_{(i,j)\in\alpha} L_1^{-i} \otimes L_2^{-j}\Big)$$
$$= \sum_{(i,j)\in\alpha} -i \underbrace{\deg L_1}_{g-1} - j \underbrace{\deg L_2}_{g-1} + 1 - g$$
$$= (1-g) \sum_{(i,j)\in\alpha} i + j + 1 = (1-g) \sum_{(i,j)\in\alpha} h_\alpha(i,j)$$

So we get that

$$Z_d^{PT} = \sum_{\alpha\vdash d} Q^{(1-g)\sum_{(i,j)\in\alpha} i+j-i} \left(\prod_{(i,j)\in\alpha} \frac{1}{(1 - Q^{h_\alpha(i,j)})^2} \right)^{1-g}$$
$$= \sum_{\alpha\vdash d} \prod_{(i,j)\in\alpha} (Q^{h(i,j)/2} - Q^{-h(i,j)/2})^{-(2-2g)}$$
$$= \sum_{\alpha\vdash d} \left(\frac{\dim_Q \alpha}{d!} \right)^{2-2g}$$

The Gromov-Witten theory is now computed via substitution $Q = e^{i\lambda}$:

$$\begin{aligned} Z_d^{GW} &= \sum_{\alpha \vdash d} \left(\prod_{\square \in \alpha} 2 \sin \frac{h(\square)\lambda}{2} \right)^{2g-2} \\ &= \sum_{\alpha \vdash d} \left(\prod_{\square \in \alpha} h_\alpha(\square)\lambda \right)^{2g-2} (1 + o(\lambda^2)) \\ &= \lambda^{d(2g-2)} \sum_{\alpha \vdash d} \left(\frac{d!}{\dim \alpha} \right)^{2g-2} + \cdots \end{aligned}$$

The last identity comes from

$$\prod_{\square \in \alpha} h_\alpha(\square) = \frac{d!}{\dim \alpha}$$

where $\dim \alpha$ is the dimension of $\alpha \vdash d$ when viewed as a representation of S_d.

So the leading term in Z_d^{GW} is

$$\lambda^{d(2g-2)} = \lambda^{2g-2}$$

where n is the genus of the domain. This corresponds to unramified covers of degree d. The number of unramified covers of degree d of C is given via

$$\sum_{\alpha \vdash d} \left(\frac{d!}{\dim \alpha} \right)^{2g-2}.$$

So for genus one, we get the familiar, $p(d)$, number of coverings of degree d. And in genus zero, we get $\frac{1}{(d!)^2} \sum_{\alpha \vdash d} (\dim \alpha)^2$. From elementary representation theory, if you take any group, and take all representations and sum the squares of their dimensions you get the order of the group, so we get that this is equal to $\frac{1}{d!}$.

The punch line is the interesting fact that while studying the Gromov-Witten invariant and deformations of curves, the same Q-deformations of integers occur, in sense of classical deformations of integers.

Appendix A

Solution to some problems

A.1 Homework set 2, Problem 3

The goal is to analyze $I_3(X, 2[\mathbb{P}^1])$ and $I_4(X, 2[\mathbb{P}^1])$ in case of $X = \mathrm{tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1)$ being the resolved conifold.

Remember that for $[I_Z] \in I_n(X, 2[\mathbb{P}^1])$, $Z \subset X$ must have its curve component be supported on $\mathbb{P}^1 \subset X$. Because

$$X \xrightarrow{p} \{xy = wz\} \subset \mathbb{C}^4$$

is a resolution of singularities. But no subscheme of this affine target space is compact. So the scheme-theoretic image of Z, $p(Z)$ has to be supported on points, and curve part of Z being supported on $p^{-1}(0)$. So Z is a length 2 thickening of $\mathbb{P}^1$ with embedded points. So for the generic fiber:

$$Z \cap \mathbb{C}^2 = \mathrm{Spec}\left(\mathbb{C}[x, y]/(x^2, xy, y^2, \alpha x + \beta y)\right)$$

So the primary component of Z is determined by a subline-bundle $L \hookrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. So $Z \subset L \subset X$ and Z is the zero section of L, hence a Cartier divisor, cut-out by exactly one equation, hecne $Z = 2[\mathbb{P}^2]$ the unique thickening of $\mathbb{P}^1$ determined by L. What is $\chi(\mathcal{O}_X)$? And what are the possible subline-bundles that can happen?

Let's say $L = \mathcal{O}(-k)$. Then $L \to \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ are given by $H^0(\mathbb{P}^1, \mathcal{O}(k-1) \oplus \mathcal{O}(k-1))$, but $H^0(\mathbb{P}^1, \mathcal{O}(k-1) \oplus \mathcal{O}(k-1)) = \mathbb{C}^{2k}$ for $k \geq 1$.

In general, let S be a smooth surface $D \subset S$. Then by Riemann-Roch

$$\chi(\mathcal{O}_D) = \int_S \mathrm{ch}(\mathcal{O}_D)\mathrm{td}(T_S).$$

By adjunction formula

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to O_D \to 0$$

and $\mathrm{ch}(\mathcal{O}_D) = 1 - e^{-D} = D - \frac{D^2}{2}$ and

$$\chi(\mathcal{O}_D) = \int_S (D - \frac{D^2}{2})(1 - \frac{K_S}{2} + \cdots) = -D\frac{K_S}{2} - \frac{D^2}{2}$$

giving the reincarnation of adjunction formula for any surface:

$$\chi(\mathcal{O}_D) = \frac{-1}{2}(D^2 + D.K_S).$$

(We need compactness, but one can always compactify the surface and then everything is supported away from the compactification locus).

We apply this to our surface $L = \mathcal{O}(-k) \subset X$. So $[\mathbb{P}^1]^2 = -k$ and $D = 2[\mathbb{P}^1]$ and $D^2 = -4k$. Also $K_S = \mathcal{O}(k) \otimes K_{\mathbb{P}^1} = \mathcal{O}(k-2)$. So $K_S.\mathbb{P}^1 = k - 2$ and $K_S.D = 2k - 4$. So

$$\chi(\mathcal{O}_D) = \frac{-1}{2}(-4k + 2k - 4) = k + 2.$$

Now $I_3(X, 2[\mathbb{P}^1])$ is the moduli of thickenings along $\mathcal{O}(-1) \hookrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-1)$

In case of $n = 4$ the space $I_4(X, 2[\mathbb{P}^1])$ has two strata of dimension 3 and 4: $\mathbb{P}^3 \supset \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{Bl}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathbb{P}^1 \times X)$ that meet in the intersection $\mathbb{P}^1 \times \mathbb{P}^1$. The $\mathbb{P}^3 = \mathbb{P}(H^0(\mathbb{P}^1, \mathcal{O}(1) \oplus \mathcal{O}(1)))$ corresponds to injections of bundles $0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and the blow up corresponds to short exact sequences $0 \to \mathcal{O}(-2) \to \mathcal{O}(-1) \to \mathcal{O}_p \to 0$.

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