Geometric Invariant Theory
Math 533 - Spring 2011

Zinovy Reichestein

Lecture notes by Pooya Ronagh

Department of mathematics, University of British Columbia
Room 121 - 1984 Mathematics Road, BC, Canada V6T 1Z2

E-mail address: `pooya@math.ubc.ca`

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Preface

The following is the lecture notes I have written from a course Professor Zinovy Reichstein taught in University of British Columbia in Winter 2011. I am thankful to him on correspondences in which he gave me the references for the material presented below which I will cite throughout and in this preface. I am also thankful to him for reading these lecture notes, providing me with comments and corrections.

What follows heavily and speacially at the beginning of the term relies on the textbook by Newstead [**4**]. A number of other sources, including papers and books by Fogarty, Mumford [**3**], Kraft, Popov and Vinberg, Deudonne and Carrell, and most recently, Birkes and Richardson have been used. Example 1.1 was provided by Dave Anderson. Chapter 2 was entirely taught by Dave Anderson as well. He mainly used [**1**] and [**2**]. The latter contains more results with non-algebraically closed fields.

CHAPTER 1

Moduli spaces

To parametrize algebraic objects of certain type, need to set up a moduli problem; i.e. a contravariant functor

$$\mathcal{F} : \mathcal{V}ar \to \mathcal{S}ets$$

where $\mathcal{F}(pt)$ = the set of objects we want to parametrize and $\mathcal{F}(S)$ is the set of families of objects over S. Note that this $\mathcal{F}$ is not uniquely determined by the objects we want to parametrize.

Recall that M is a fine moduli space if $\mathcal{F}$ is represented by M, i.e. $\mathcal{F}(S) = \mathrm{Mor}(S, M)$ for any variety S. Also recall that existence of a fine moduli space M is equivalent to existence of a universal object above $X \mapsto M$ where by a universal object we mean

$$\mathcal{F}(S) = \{\mathcal{F}(\eta)(X) : \eta : S \to M\}$$

is satisfied for any S.

Remark. Note that existence of a nontrivial fibration Y over some S implies that the fine moduli space does not exist.

Definition 1. A variety M is called a coarse moduli space for $\mathcal{F}$ if

(1) $\mathcal{F}(pt) \cong M$ (as a set).

(2) $\mathcal{F} \to \mathrm{Mor}(-, M)$ is a morphism of functors.

(3) Any other morphism of functors $\mathcal{F} \to \mathrm{Mor}(-, N)$ factors (uniquely; already follows since it is already given on points) through some morphism $M \to N$ of varieties.

Remark. A fine moduli space is a coarse moduli space.

1. An example

Let V be an n-dimensional vector space, or simply k^n. The objects we want to classify are linear maps $V \xrightarrow{T} V$. We say two such objects $T : V \to V$ and $G' : V \to V$ are equivalent if there is $g \in \mathrm{GL}(V)$ such that

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ \downarrow g & & \downarrow g \\ V & \xrightarrow{T'} & V \end{array}$$

or that T and T' are conjugate. The moduli problem will be

$$\mathcal{F}(S) = \{(E,T) : E \to S \text{ vector bundle of rank } n\ ,\ \begin{tikzcd} E \arrow[r, "T"] \arrow[dr] & E \arrow[d] \\ & S \end{tikzcd} \}/\cong .$$

Here isomorphism classes are given by commutative diagrams

$$\begin{tikzcd} V \arrow[r, "T_1"] \arrow[d, "g"'] & V \arrow[d, "g"] \\ V \arrow[r, "T_2"] & V \end{tikzcd} .$$

We will see that

$$\begin{aligned} \text{All endomorphisms} &\to \text{No moduli space} \\ \text{semisimple endomorphisms} &\to \text{coarse moduli space} \\ \text{cyclic endomorphisms} &\to \text{Fine moduli space} \end{aligned}$$

PROPOSITION 1. *There is no fine moduli space for this functor.*

PROOF. Consider $E = \mathbb{A}^1 \times V$ over $S = \mathbb{A}^1$ and let $T : E \to E$ be given via

$$(t,v) \mapsto (t, \begin{pmatrix} 0 & t & 0 & \cdots \\ 0 & 0 & t & \cdots \\ & & \ddots & \\ 0 & \cdots & 0 & t \\ 0 & 0 & \cdots & 0 \end{pmatrix} v)$$

For all $t \neq 0$ these matrices are equivalent by conjugation by $\mathrm{diag}(1,t,t^2,\cdots,t^{n-1})$ and the matrix $T|_{t=0}$ is not equivalent to them. □

1.1. Cyclic endomorphisms. $T : V \to V$ is cyclic if $v, Tv, \cdots, T^{n-1}v$ form a basis of V for some $v \in V$. There would be two ways of defining the families here: (1) $T : E \to E$ over S is locally cyclic if T_s is cyclic for any $s \in S$. (2) To require that T is globally cyclic.

1.2. Semisimple endormophisms.

$$\mathcal{F}(pt) = \{ \text{ semisimple endomorphisms } T : V \to V \ \}$$

where semisimple means diagonalizable, and let's say $\mathrm{char}(k) \nmid n$ and

$$\mathcal{F}(S) = \{(E,T) : E \to S \text{ vector bundle}\ ,\ \begin{tikzcd} E \arrow[r, "T"] \arrow[dr] & E \arrow[d] \\ & S \end{tikzcd} \text{ semisimple endomorphisms } \}/\cong$$

THEOREM 1.1. *(1) $M = \mathbb{A}^n$ is a coarse moduli space for $\mathcal{F}$. here the isomorphism between $\mathcal{F}(pt)$ and $\mathbb{A}^n$ is given by*

$$T \overset{charac.poly}{\longrightarrow} (c_1, \cdots, c_n)$$

where $x^n + c_1 x^{n-1} + \cdots + c_n$ is the characteristic polynomial .

(2) There is not fine moduli space for $\mathcal{F}$.

PROOF. For (1) need to check conditions (2) and (3) of the definition. Given a family of endomorphisms

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr] & & E \arrow[dl] \\ & S & \end{tikzcd}$$

over S, want to check that the induced map $f : S \overset{char}{\longrightarrow} \mathbb{A}^n$ is a morphism. It is enough to do this locally on S: Cover S by affine opens $\{S_i\}$ where $E \cong S_i \times V_j$ is the trivial bundle over S_i. Over S_i choose a global basis, $b_1, \cdots, b_n$ of global sections for E. In this basis T is given by an $n \times n$ matrix with entries in $k[S_i]$. Taking the characteristic polynomial of this matrix we see that $f|_{S_i}$ is a morphism. To check condition (3) suppose $\mathcal{F} \to \mathrm{Mor}(-, N)$ is a morphism of functors. Want to show this factors through a morphism $M \to N$. Consider the following family: $S = \mathbb{A}^n$ and $E = \mathbb{A}^n \times V$. Fix a basis $(\lambda_1, \cdots, \lambda_n)$ for k^n and assume

$$T = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{pmatrix}$$

and apply our morphism to this family.

$$\begin{tikzcd} E \arrow[d] & & \\ S = \mathbb{A}^n \arrow[r, "S_n\text{equiv}"] \arrow[d, "\lambda=(\lambda_1,\cdots,\lambda_n)\mapsto(e(\lambda),\cdots,e(\lambda))"] & N \arrow[r, "\alpha_i"] & k \\ M & & \end{tikzcd}$$

where $e_i = ($ the i-th characteristic polynomial $)(-1)^i$.

Case 1: N is affine. Choose generators $\alpha_1, \cdots, \alpha_n \in k[N]$. Pull them back to S. The pull-backs are symmetric functions on S; i.e. polynomials in $e_1(\lambda), \cdots, e_n(\lambda_n)$. Therefore factors through M.

Case 2: General N. We consider the special case of the natural action of $G = S_n$ on $X = A^n$.[1] In general cover N by affines and use the fact that $e_1(\lambda), \cdots, e_n(\lambda)$ generate the field of S_n-invariant rational functions on $S = \mathbb{A}^n$. (Check!)

For second part assume the contrary that $M = \mathbb{A}^n$ is a fine moduli space (by uniqueness of coarse moduli space). Then in particular M carries a universal family; i.e. there is a vector bundle $E \to \mathbb{A}^n$ and a semisimple endomorphism $T : E \to E$ over $\mathbb{A}^n$ such that the characteristic polynomial of T is $x^n + a_1 x^{n-1} + \cdots + a_n$ where $a_1, \cdots, a_n$ are the coordinate functions on $\mathbb{A}^n$. We claim that such a family cannot exist. Choose $g \in k[t_1, \cdots, t_n]$ such that E is trivial over the principal open subset $U = \{g \neq 0\}$ in $\mathbb{A}^n$ and $0 \in U$ i.e. $g(0) \neq 0$. Over U choose a basis of global sections and write the matrix as $(f_{ij})_{n \times n}$ where $f_{ij} \in k[U] = k[a_1, \cdots, a_n, g^{-1}]$. On the other hand T_0 is nilpotent and semisimple, hence zero. Thus $f_{ij}(0) = 0$ for all i, j. Thus $\det(T)$ vanishes to order n at zero. But $\det T = (-1)^n a_n$. By our assumption on $\mathrm{char}(T)$ if $n \geq 2$, this is a contradiction. □

Question: What happens if $n = 1$? (Hint: consider the way we introduced families for cyclic endomorphisms. The answer depends on specific function $\mathcal{F}$ we consider, i.e. the type of families of semisimple endomorphisms we allow over a given bases.)

Example 1.1. Look at the moduli problem $\mathcal{M}_{1,1}$ of families of 1 pointed elliptic curves mod isomorphisms.[2] Claim:

(1) $M_{1,1} \cong \mathbb{A}^1$ is the coarse moduli space (Use j-invariants machinery - See Hartshorne).

(2) There is no fine moduli space; there is a family $\mathfrak{X} \to S$ with all fibers isomorphic but tnontrivial. Let E be an elliptic curve $\Gamma \subseteq E \times E \times E$. And $\Gamma = \{(x, y, x + y)\}$. And look at first projection.

▸ Exercise 1. Show this is NOT an example (that the fibration is nontrivial).

▸ Exercise 2. Fix an elliptic curve E, and look at the fibration over $\mathbb{P}^1 \times_{0,\infty} \mathbb{P}^1$ given by $E \times \mathbb{P}^1 \cup_{E \times \{0\}, E \times \{\infty\}} E \times \mathbb{P}^1$ an gluing on zero via identity and on ∞ via inverse. This has only 4 sections, and therefore is nontrivial.

⌟

[1] We have supplied a proof of this case in the course of proving that $\mathcal{S}pec\, k[X]^G$ is a categorical quotient for the G-action on X, where G is linearly reductive group, acting on an affine space X.

[2] This example was given by David Anderson, at the beginning of his class.

CHAPTER 2

Crash course on linear algebraic groups

DEFINITION 2. An algebraic group is a group object in the category of varieties; i.e. G is a variety and morphisms are $G \times G \to G$ and $G \to G$ and has a point $e \in G$. In the schemey language, the functor $h_G : \mathcal{S}ch/k \to \mathcal{S}et$ factors though the category of groups.

DEFINITION 3. A linear algebraic group is an algebraic group G that is an affine variety.

Right away this tells you something about the coordinate ring of it! But let's do an example first:

Example 0.2.

$$\mathbb{G}_m = k^* = \mathcal{S}pec(k[x, x^{-1}])$$

. ⌟

Note: The coordinate rings become $k[G]$ have the structure of a commutative Hopf algebra: $\delta : k[G] \to k[G] \otimes k[G]$ the co-multiplication of the the algebra and $c : k[G] \to k[G]$ is the antipade map and there is also a counit $\varepsilon : k[G] \to k$.

Example 0.3. For the multiplicative group:

$$\delta : k[x, x^{-1}] \to k[x, x^{-1}] \otimes k[x, x^{-1}]$$
$$x \mapsto x \otimes x$$

and

$$c : k[x, x^{-1}] \to k[x, x^{-1}]$$
$$x \mapsto x^{-1}$$

and $\varepsilon : x \mapsto 1$. ⌟

▸ EXERCISE 3. Work out δ, c, ε for $\mathbb{G}_a$ ($\mathbb{G}_a = k$ with multiplication).

▸ EXERCISE 4. $\mathrm{Gl}_n = \mathcal{S}pec\, k[x_{ij}]_{\det}$.

▸ EXERCISE 5. Any Zariski closed subgroup of Gl_n; O_n and Sl_n and $\mathbb{B}_n \subset \mathrm{Gl}_n$ the subgroup of upper-triangular matrices (called the Borel subgroups). $\mathbb{D}_n \subseteq \mathbb{B}_n \subset \mathrm{Gl}_n$ the subgroup of diagonal matrices (maximal torus) and $\mathbb{U}_n \subset \mathbb{B}_n \subset \mathrm{GL}_n$ strictly upper-triganluar matrices (1's on the diagonal) (unipotent subgroup).

PROPOSITION 2. *Every linear algebraic group can be embedded in some* Gl_n*. The slogan is that affine* $\Leftrightarrow$ *linear.*

DEFINITION 4. A (rational) representation of G is a homomorphism $G \to \mathrm{Gl}(V)$ where V is a vector space (homomorphism of algebraic groups is a map of the varieties that is also a map of groups).

A representation V is an irreducible one if there is no $0 \subsetneq W \subsetneq V$ with $G.W \subseteq W$.

REMARK. You can also talk about representations where V is infinite. And there is an important example of it. But always assume locally finite: i.e. for $v \in V$ there is a fintie dimenional W where $v \in W \subsetneq V$ which is G-stable ($G.W \subseteq W$).

Example 0.4. G acts on $k[G]$ by $g.f(x) = f(g^{-1}x)$ for all $x \in G$.[1] ⌟

LEMMA 1. *If* $W \subsetneq k[G]$ *is any finite dimensional subspace, then there is a subspace* $W \subseteq V \subseteq k[G]$ *such that* V *is finite-dimensional and* $G.V = V$*.*

PROOF. Without loss of generality assume $W = \mathrm{span}\{f\}$. Write $\widetilde{\delta}(f) = \sum_i m_i \otimes f_i \in k[G \times G]$. $\widetilde{\delta}$ corresponds to $G \times G \to G$ via $(g_1, g_2) \mapsto (g_1^{-1} g_2)$. Then for every $g \in G$

$$(g.f)(x) = f(g^{-1}x) = \sum_i m_i(g) f_i(x)$$

So $g.f = \sum_i m_i(g) f_i$. So $g.f \in \mathrm{span}\{f_1, \cdots, f_n\}$. Then $\mathrm{span}\{g.f : g \in G\}$ does the trick. □

PROOF OF PROPOSITION. Take generators $f_1, \cdots, f_n$ of $k[G]$. By lemma assume they are a basis for a G-stable subspace of $k[G]$. As in the lemma have $m_{ij} \in k[G]$ such that $g.f_i = \sum_j m_{ij}(g) f_j$. and the map is $k[x_{ij}]_{\det} \to k[G]$ via $x_{ij} \mapsto m_{ij}$. And it remains to show that this is surjective. Because

$$f_i(g^{-1}) = f_i(g^{-1}e) = \sum_j m_{ij}(g) f_j(e)$$

see $f_i = \sum_j f_j(e) m_{ij}$ and m_{ij}'s generate $k[G]$ so the map is surjective. □

1. Diagonalizable groups and characters

$\mathbb{D}_n \subseteq \mathrm{GL}_n$ the group of diagonal matrices. Note that $k[\mathbb{D}_n] \cong k[x_1^{\pm}, \cdots, x_n^{\pm}]$ which is isomorphic to the group algebra $k[\mathbb{Z}^n]$.

DEFINITION 5. A character of an algebraic group G is a homomorphism $\chi : G \to \mathbb{G}_m$.

[1] The author has been struggling to figure this out in the language of groups schemes!

The set of characters

$$X(G) = \mathrm{Hom}_{alg.gp.}(G, \mathbb{G}_m)$$

is an abelian group under pointwise multiplication. (Warning: people intend to write the operation of this group additively sometimes!)

$$\chi_1.\chi_2(g) = \chi_1(g)\chi_2(g).$$

Example 1.1. $X(\mathbb{G}_m) = \mathbb{Z}$ via $\mathrm{id} \mapsto 1$ and a character of $\mathbb{G}_m$ is given via $\chi : z \mapsto z^n$ in general. ⌟

Note that $\mathbb{D}_n \cong (\mathbb{G}_m)^n$ and hence $\chi(\mathbb{D}_n) \cong \mathbb{Z}^n$ (in of course a non-canonical way) and therefore another way of writing the isomorphism in the beginning is $k[\mathbb{D}_n] = k[X(\mathbb{D}_n)]$. Moral: $\mathbb{D}_n$ has lots of characters in contrast to Sl_n which has no nontrivial characters.

Definition 6. A linear algebraic group is diagonalizable if its isomorphic to a closed subgroup of some $\mathbb{D}_n$ and if it also happens to be connected then it is called a torus.

Diagonalizable groups are very useful for the following structure theorem:

Proposition 3. *For a linear algebraic group D the following are equivalent:*

(1) D is diagonalizable.

(2) $X(D)$ is finitely generated abelian group and $k[D] \cong k[X(D)]$.

(3) Every (finite dimensional, rational) representation of D is isomorphic to a direct sum of 1-dimensional representations.

(4) $D \cong (k^*)^r \times A$ for some finite abelian group A.

Corollary 1. *T is a torus iff $T \cong (k^*)^r$ (assuming algebraically closedness). ($\mathcal{S}pec(\mathbb{F}_3[x])_{(x)}$ is not connected.)*

Remark. In (3) if a given representation V you can write $V = \oplus_{\chi \in X(D)} V_\chi$ where

$$V_\chi = \{v \in V : g.v = \chi(g)v, \forall g \in D\}$$

and these latter are called weight spaces.

Remark. In (4) we certainly need the condition that A has no p-torsion when $\mathrm{char}(k) = p$.

Remark. These tori are essentially the only groups with semi-simple representation theory independent of character of the field (linearly reductive condition).

Example 1.2 (of (3))**.** $T = (k^*)^2 \subseteq \mathrm{Gl}_2$ and acting on $M_{2\times 2}$ by conjugation. Then $T = \{\begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}\}$ is generated as an abelian group (has basis as Z-module) given by

$$\chi_1(g) = z_1, \quad \chi_2(g) = z_2.$$

Note that $g.\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & z_1 z_2^{-1} b \\ z_1^{-1} z_2 c & d \end{pmatrix}$ and

$$M_{2\times 2} = k.\overbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}^{\chi_1\chi_2^{-1}} \oplus k.\overbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}^{\chi_1^{-1}\chi_2} \oplus k.\overbrace{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}}^{0}.$$

of weights $\chi_1\chi_2^{-1}$ and $\chi_1^{-1}\chi_2$ and 0. ⌟

Example 1.3 (of (4))**.** $X(D) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ then if we work backwards to find the algebraic group itself then

$$k[X(D)] \cong k[x, x^{-1}, y]/(y^2 - 1)$$

shows that $D = \mathcal{Spec}\, k[X(D)] = \mathbb{G}_m \times \mu_2$. Note that μ_2 is not reduced if $\mathrm{char}(k) = 2$. ⌟

Proof. See Springer. We leave 2 ⇒4 ⇒1 ⇒2 and 3 ⇒1 as exercise. Remains to prove 2 ⇒3. Given a representation $D \xrightarrow{\varphi} \mathrm{GL}(V)$ choose a basis for V so

$$\varphi(g) = (a_{ij}(g))$$

for some $a_{ij} \in k[D] = k[X(D)]$ so

$$a_{ij} = \sum_{\chi} c_{ij}^{\chi}\chi.$$

Write $\varphi(g) = \sum_\chi \chi(g) A_\chi$ where $A_\chi = (c_{ij}^{\chi})$. Will show $A_\chi \in \mathrm{End}(V)$ is projection onto V_χ.

Claim: For $\chi, \psi \in X(D)$ have

$$A\chi A_\psi = \delta_{\chi,\psi} A_\chi \text{ (orthogonal independence).}$$

In fact since $\varphi(gh) = \varphi(g)\varphi(h)$ get

$$\sum_{\eta \in X(D)} \eta(gh) A_\eta = \sum_B \left(\sum_{A_\chi A_\psi = B} \chi(g)\psi(h) \right) B.$$

So entry-wise

$$\sum_\eta c_{ij}^{\eta} \eta(g)\eta(h) = \sum_{\chi,\psi} b_{ij}^{\chi,\psi} \chi(g)\psi(h).$$

We now use the theorem of linear independence of character for the group $D \times D$ and get

$$b_{ij}^{\chi,\psi} = c_{ij}^{\eta} \delta_{\chi,\psi,\eta}.$$

Finally

$$\sum_{\chi \in X(D)} A_\chi = 1$$

and this is because the left hand side is $\sum_\chi \chi(1) A_\chi = \varphi(1)$. So $V = \oplus V_\chi$ where $V_\chi = \mathrm{im}(A_\chi)$. □

Dual to characters: Called the co-characters also, are define by

DEFINITION 7. For an algebriac group G a one-parameter subgroup is $\lambda : \mathbb{G}_m \to G$. $Y(G) = \mathrm{Hom}_{al.gp.}(\mathbb{G}_m, G)$ is also a group.

There is always a natural pairing

$$X(G) \times Y(G) \to \mathbb{Z}$$

via $(\chi, \lambda) \mapsto \chi \circ \lambda \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$.

▸ EXERCISE 6. When G is a torus this is a perfect pairing (i.e. $Y(G) = \mathrm{Hom}_{\mathbb{Z}}(X(G), \mathbb{Z})$).

2. Reductive groups

Let $k = \overline{k}$ and G linear algebraic group. An element $x \in G$ is semisimple if there is a faithful (injective) representation $\varphi : G \to \mathrm{Gl}_n$ such that

$$\rho(x) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}_{n \times n}.$$

$x \in G$ is unipotent if there is $\rho : G \to \mathrm{Gl}_n$ such that $\rho(x) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

THEOREM 2.1 (Jordan decomposition). *For any $x \in G$ there are unique elements $x_s, x_u \in G$ such that*

$$x = x_s x_u = x_u x_s$$

and x_s is semisimple and x_u is unipotent. Moreover, any homomorphism $\varphi : G \to H$ preserves ss + unipotent parts, i.e. $\varphi(x)_s = \varphi(x_s)$ and $\varphi(x)_u = \varphi(x_u)$.

PROOF. See Humphreys, section 15.3. □

DEFINITION 8. G is unipotent if all $x \in G$ are unipotent.

REMARK. Any image or subgroup of a unipotent G is unipotent.

DEFINITION 9. G is solvable if $G = G_0 \supseteq G_1 \supseteq \cdots \supseteq$ terminates in $\{e\}$, where $G_i = [G_{i-1}, G_{i-1}]$ is the commutator group.

Note: For any algebraic group G, (G, G) is generated by all $ghg^{-1}h^{-1}$ and it is a closed subgroup (not obvious).

Example 2.1. $\mathbb{U}_n \subseteq \mathrm{GL}_n$ the subgroup of $\begin{pmatrix} 1 & & * \\ 0 & * & 1 \end{pmatrix}$ elements is unipotent. It is also solvable:

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \supset \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \supset \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

⌟

Example 2.2. $B_n\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$ is solvable because $(\mathbb{B},\mathbb{B}) = \mathbb{U}$. ⌟

REMARK. Any subgroup of a solvable group is solvable.

Generally unipotent group is solvable. The reason is

PROPOSITION 4 (Lie - Kelchin). *(char = 0?) If G is unipotent, then for any representation $\rho : G \to \mathrm{Gl}(V)$ there's a basis of V such that $\rho(G) \subseteq \mathbb{U}_n$. Similarly if G is solvable, then for any representation $\rho : G \to \mathrm{GL}(V)$ there is a basis of V such that $\rho(G) \subseteq \mathbb{B}_n$.*

3. Borel subgroups

DEFINITION 10. A Borel subgroup of G is a maximal connected solvable (closed) subgroup.

Example 3.1. $\mathbb{B} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq \mathrm{GL}_n$. ⌟

THEOREM 3.1 (Humphrey, 21.3). *All Borel subgroups in G are conjugate.*

PROOF. Uses

(1)
$$G/B \cong \{ \text{ Borel subgroups } \}$$
is a projective variety.

(2) Borel's fixed point theorem. (An action of a solvable group in a projective variety has a fixed point). This uses Lie-Kelchin.

□

COROLLARY 2. *All maximal tori in G are conjugate.*

The reason is that if T is solvable then $T \subseteq B$. Assume G is connected and nontrivial ($\neq \{e\}$. Then the radical of G is the maixmal connected solvable normal subgroup of G, $R(G)$. Adding the word normal makes it unique! The unipotent radical is the maximal connected unipotent normal subgroup of G, written as $R_u(G)$.

REMARK. $R_u(G) \subseteq R(G)$.

Example 3.2. $G = \mathrm{GL}_n$ then $R(\mathrm{GL}_n) = k^* = \{$scalara matrices $\}$.
$$1 \to k^* \to \mathrm{Gl}_n \to PGL_n \to 1$$
where the last term is a simple group. In particular from the above remark $R_u(G) = 1$ here. ⌟

Example 3.3. $P = \{\begin{pmatrix} *_{2\times 2} & *_{2\times 2} \\ 0 & *_{2\times 2} \end{pmatrix}\} \subseteq \mathrm{Gl}_4$ has

$$R_u(P) = \{\begin{pmatrix} I_{2\times 2} & * \\ 0 & I_{2\times 2} \end{pmatrix}\}.$$

and

$$R(P) = \{\begin{pmatrix} * & 0 & * & * \\ 0 & * & * & * \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}\}.$$

⌟

DEFINITION 11. G is semisimple if $R(G) = \{e\}$. G is reductive if $R_u(G) = \{e\}$.

REMARK. ss ⇒reductive.

Example 3.4. $\mathrm{Sl}_n, PGL_n, \mathrm{Sl}_n \times \mathrm{Sl}_m$ are ss, and GL_n, T are reductive. ⌟

REMARK. $Z(\mathrm{Sl}_n)\mu_n$ is not connected.

REMARK. If G is semisimple, then $Z(G)$ is finite. Otherwise, $Z(G)^\circ$ is a nontrivial solvable normal subgroup.

REMARK. If G is reductive then $Z(G)^\circ = R(G)$ is a torus and (G,G) is semisimple.

Example 3.5. $(\mathrm{GL}_n, \mathrm{GL}_n) = \mathrm{Sl}_n$. ⌟

Example 3.6. But $R(\mathbb{B}) = \mathbb{B}) \neq k^* = Z(\mathbb{B})$. ⌟

REMARK. Any G, $G/R(G)$ is semisimple and $G/R_u(G)$ is reductive.

Example 3.7. $\mathrm{Gl}_n / k^* = PGL_n$ and $\mathbb{B}/\mathbb{U} = \mathbb{D}$. ⌟

Up to finite quotient, semisimple groups are calassified to be

$$\mathrm{Sl}_n, \mathrm{SO}_{2n+1}, Sp_{2n}, \mathrm{SO}_{2n}$$

and the finite ones are G_2, F_4, E_6, E_7, E_8.

CHAPTER 3

Action of algebraic groups

Let $\varphi : G \times X \to X$ be the action (morphism) of an algebraic group G on variety X. An orbit $G.x$ of x is $\varphi(G \times \{x\})$ and the stabilizer is $\mathrm{Stab}_G(x) = \{g \in G : g.x = x\}$ which is a closed subgroup of G.

Example 0.8. Any linear representation $G \to \mathrm{GL}(V)$ gives rise to a (linear) action of G on V and an action of G on $\mathrm{Gr}(n,d)$ where $n = \dim V$ and $1 \leq d \leq n-1$. (Check that this is an algebraic action). ⌟

Definition 12. Given the action of an algebraic group G on an algebraic variety X, we say that a morphism $\pi : X \to Q$ is a categorical quotient for this action if it satisfies the universal property:

(1) π is constant on orbits, i.e. $\pi(g.x) = \pi(x)$ for all $x \in X$ and $g \in G$.

(2) $\pi' : X \to Q'$ is another morphism constant on orbits then π' factors through π

$$\begin{tikzcd} X \arrow[d, "\pi"] \arrow[dr, "\pi'"] & \\ Q \arrow[r, "!"] & Q' \end{tikzcd}$$

Example 0.9. The natural (permutation) action of $G = S_n$ on $\mathbb{A}^n$. Here $\pi : \mathbb{A}^n \to \mathbb{A}^n$ is the categorical quotient. ⌟

Remark. If the categorical quotient exists, it is unique.

1. Relation of coarse moduli spaces and categorical quotients

Let $\mathcal{F}$ be a moduli functor.

Definition 13. A family X over S (i.e. an element of $\mathcal{F}(S)$) has *a local universal property* if for any family Y over T and any point $t \in T$ there exists a Zariski open neighborhood U of t in T and a morphism $f : U \to S$ so that $Y|_U$ is the pullback of X via f.

Note: f may not be unique.

Example 1.1.

$$\mathcal{F}(S) = \{(E,T) : E \text{ vector bundle over } S\ , T : E \to E \text{ endomorphism over } S\ \}$$

$$\begin{tikzcd} E = M_n \times V \arrow[r, "T"] \arrow[d] & M_n \times V \arrow[dl] \\ S = M_n & \end{tikzcd}$$

with $T(A,v) = (A, Av)$ for a generic matrix $T = \left(x_{ij}\right)$ where x_{ij} are coordinate functions on M_n. Indeed if

$$\begin{tikzcd} E' \arrow[r, "T'"] \arrow[d] & E' \arrow[dl] \\ S' & \end{tikzcd}$$

is another family of endomorphism sand $t \in S$ can trivialize E in a neighborhood of t over U, T' is given by a matrix $\left(f_{ij}\right)$ where $f_{ij} \in k[M_n]$. Here we assume S' affine. The desire map $f : U \to S = M_n$ is given by

$$u \mapsto \left(f_{ij}(u)\right).$$

⌟

Remark. Hilbert's Theorem 90 shows that a vector bundle (trivial in etale topology) is Zariski trivial.

Theorem 1.1. *Suppose a family X over S has a local universal property for $\mathcal{F}$. Moreover, assume that an algebraic group G is acting on S so that for $s,t \in S$, $X_s \cong X_t$ iff s,t are in the same G-orbits. Then*

(1) *Morphisms of functors $\mathcal{F} \to \mathrm{Mor}(-,N)$ are in a natural bijective correspondence with morphisms of varieties $S \to N$, constant on orbits.*

(2) *If M is a coarse moduli space for $\mathcal{F}$ then M is a categorical quotient for the G-action on S with quotient map induced by $\mathcal{F}(S) \to \mathrm{Mor}(S,M)$ as the image $X \mapsto \pi : S \to M$.*

(3) *Conversely suppose the G-action on S admits a categorical quotient $\pi : S \to Q$ then (Q,π) is a coarse moduli space for $\mathcal{F}$ if and only if Q is an "orbit space", i.e. G-orbits in S are precisely the fibers of π.*

Example 1.2. If $S = M_n$ and $G = \mathrm{Gl}_n$ acting on S by conjugation as in previous example, we know that

$$\begin{pmatrix} 0 & 1 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

does not have a closed orbit, since the 0 matrix lies in the closure. Thus cannot have an orbit space for this action. ⌟

PROOF. (1) Suppose we are given $\mathcal{F} \to \mathrm{Mor}(-.N)$. The image of $X \in \mathcal{F}(S)$ is a morphism $S \to N$. This is constant on orbits since any two equivalence X_s, X_t are mapped to the same point in N. Conversely, suppose a morphism $f : S \to N$ constant on orbits is given. We want to use it to define a morphism $\alpha : T \to N$ for any family Y over T. Define α locally on T: Cover T by opens U_i such that over each U_i, $Y|_{U_i}$ is obtained from X via a morphism $\beta_i : U_i \to S$ (this is our local universal property). Then define $\alpha_i : U_i \to N$ as the composition of $\beta_i : U_i \to S$ and $f : S \to N$. Now we check that α_i and α_j coincide on $U_i \cap U_j$. This will allow us to patch them together to a single map $\alpha : T \to N$. In fact, $\alpha_i(t) = \alpha_j(t)$ for any $t \in U_i \cap U_j$ because $Y_t \cong X_{\beta_i(t)} \cong X_{\beta_j(t)}$ so $\beta_j(t)$ and $\beta_i(t)$ are mapped via f to the same point in N. Check that the resulting maps $\alpha : T \to N$ give rise to a morphism of functors $\mathcal{F} \to \mathrm{Mor}(-, N)$.

(2) Since M is a coarse moduli space for $\mathcal{F}$ by the condition of local universal property and the correspondence of isomorphism classes of fibers of X and G-orbits it is clear the π sends each G-orbit in S to a point in M, i.e. is constant on G-orbits. If $f : S \to N$ is constant on orbits, then we want to show that f factors through π. This follows form (1) and the definition of coarse moduli space.

(3) To be a coarse moduli space, Q has to be in one-to-one correspondence with $\mathcal{F}(pt)$ which is G/S, the set of G-orbits. Thus if Q is a coarse moduli space then it is an orbit space. The rest follows from (1).

□

Example 1.3. Let $S = \mathbb{A}^n$ and $G = S_n$, acting by permutations. This is related to the functor of semisimple endomorphisms. The categorical quotient $\pi : \mathbb{A}^n \to \mathbb{A}^n$ given by $(x_1, \cdots, x_n) \to (t - x_1)\cdots(t - x_n)$ is the orbit space (Generally the categorical quotient for any action of a finite group on an affine variety is always an orbit space. In this case every point is properly stable in the sense which will defined later. ⌟

CHAPTER 4

Categorical quotient of action of algebraic groups on affine spaces

Now we want to construct the categorical quotient for a given action of an algebraic group G on an algebraic variety X. Assume for now that X is affine.

Observation 1: Suppose $\pi : X \to Y$ is a categorical quotient and Y is affine. The universal property says that the image of

$$\pi^* : k[Y] \to k[X]$$

is $k[X]^G$ = the ring of invariant functions on X. By universal property π is dominant (Check!). So π^* is injective. We conclude that π is induced by the inclusion $k[X]^G \hookrightarrow k[X]$.

Conclusion: If π exists and Y is affine then π^* is the inclusion of $k[Y] = k[X]^G$ in $k[X]$ and this should in particular be a finitely generated k-algebra. (Hilbert's 14-th problem - Nagata and Weyl: Not true in general but if G is reductive then this is the case.)

Here is an example when $k[X]^G$ is finitely generated but $\pi : X \to Y$ is not the categorical quotient.

Example 0.4. $X = \mathrm{Gl}_n$ and G is a subgroup of Gl_n consisting of matrices of the form

$$g = \begin{pmatrix} A_{a\times a} & 0 \\ B_{b\times a} & C_{b\times b} \end{pmatrix}$$

where $n = a + b$ is a fixed partition of n. Then G acts on $X = \mathrm{Gl}_n$ by multiplication on the left $g.M = gM$. Let $\pi : X \to \mathrm{Gr}(n, a)$ be the mapping

$$M \mapsto \text{ span of first } a \text{ rows of } M$$

Observe that the fibers of π are precisely the G-orbits in X. Now π is the categorical quotient for the G-action on X: say $X \to N$ is constant on G-orbits, then on the level of points we get a map $\psi : \mathrm{Gr}(n, a) \to N$. To show that ψ is a morphism consider the standard affine patching of $\mathrm{Gr}(n, a)$. We construct a section $s_{i_1,\cdots,i_a}$ of π on each $U_{i_1,\cdots,i_a}$. Say for simplicity $(i_1,\cdots,i_a)=(1,\cdots,a)$. Then the section is

$$\begin{pmatrix}1&0&\cdots&0&a_{1,a+1}&\cdots&a_{1,n}\\0&1&\cdots&0&a_{2,a+1}&\cdots&a_{2,n}\\&&\ddots&&&&\\0&0&\cdots&1&a_{a,a+1}&\cdots&a_{a,n}\end{pmatrix}\mapsto\begin{pmatrix}1&0&\cdots&0&a_{1,a+1}&\cdots&a_{1,n}\\0&1&\cdots&0&a_{2,a+1}&\cdots&a_{2,n}\\&&\ddots&&&&\\0&0&\cdots&1&a_{a,a+1}&\cdots&a_{a,n}\\0&0&\cdots&0&1&&\\&&&&&\ddots&\\0&0&\cdots&0&0&&1\end{pmatrix}$$

But $k[X]^G=k[\mathrm{Gr}(n,a)]=k$ finitely generated but there is no affine quotient. ⌟

1. Hilbert's 14-th problem

Let X be affine, and G act on X. Is $k[X]^G$ a finitely generated k-algebra?

We'll show that the answer is affirmative if G is reductive. The special case of this is $G=\mathrm{Sl}_2$ acting on $\mathbb{A}^{n+1}$ = the space of binary forms of degree n. SL_2 acts by coordinate changes:

$$f(x,y)=a_0x^n+\cdots+a_ny^n\sim(a_0,\cdots,a_n)\in\mathbb{A}^{n+1}\overset{g\cdot}{\mapsto}f((x,y)g^{-1}).$$

The question is to find $k[\mathbb{A}^n]^{\mathrm{Sl}_2}$ (for instance, find the *fundamental invariant*, i.e. the generators of the algebraic of invariants). The question of computing the fundamental invariants is still open. Known cases are:

	n	ring of invariants
	1	$k[a_0,a_1]^{\mathrm{Sl}_2}=k$
	2	$k[a_0,a_1,a_2]^{\mathrm{Sl}_2}=k[D]:D=$ discriminant $=a_1^2-4a_0a_2$
	3	$k[a_0,a_1,a_2,a_3]^{\mathrm{Sl}_2}=k[\Delta]:\Delta=$ discriminant ,$\deg\Delta=4)$
	4	$k[a_0,a_1,a_2,a_3,a_4]^{\mathrm{Sl}_2}=k[f_2,f_3]:\deg f_i=i$
Sylvester	5	4 fundamental invariants of degrees 4, 8, 12, 18
	6	5 fundamental invariants of degrees 2, 4, 6, 10, 15
1986	7	29 fundamental invariants of degrees 4 (6), 8 (3), 12 (6), 14 (4), 16 (2), 18 (9), 20, 22, 26, 30
1967	8	9 invariants of degrees 2, 3, 4, 5, 6, 7, 8, 9, 10
2010	9	92 fundamental invariants
	10	106 fundamental invariants

What matters from the geometric point of view is that the embedding $k[a_0,\cdots,a_n]^{\mathrm{Sl}_2}\hookrightarrow k[a_0,\cdots,a_n]$ gives rise to a morphism

$$\begin{tikzcd} a\in \arrow[d, mapsto] & \mathbb{A}^{n+1} \arrow[d, "\pi"] \\ (f_1(a),\cdots,f_N(a))\in & Q\subseteq\mathbb{A}^N \end{tikzcd}$$

where $f_0, \cdots, f_N$ are the generators of $k[a_0, \cdots, a_n]^{\mathrm{Sl}_2}$.

LEMMA 2. $N \geq \dim T_{\pi(0)}(Q)$.

PROOF. Let M be the maximal ideal of $\pi(0)$ in $k[a_0, \cdots, a_n]^{\mathrm{Sl}_2}$. Suppose $f_1, \cdots, f_N$ generate $k[Q]$ as a k-algebra. After subtracting $f(\pi(0))$ from f_i we may assume each $f_i \in M$. Then $\overline{f_1}, \cdots, \overline{f_N}$ generate M/M^2 as a k-vector space. □

Using this lemma, V. Kac showed that $N \geq p(n-2)$ where $p(x)$ = number of partition of x. Hardy and Ramanujan showed

$$p(x) \sim \frac{e^{\sqrt{2\pi x/3}}}{4\sqrt{3}x}.$$

The general results are:

THEOREM 1.1 (Gordan, 1868). *$k[a_1, \cdots, a_n]^{\mathrm{Sl}_2}$ is finitely generated.*

THEOREM 1.2 (Hilbert, 1890). *Nullestellensatz and Hilbert's basis theorem $\Rightarrow$a new non-constructive proof.*

The following theorem will motivate the next section.

THEOREM 1.3. *If G is a linearly reductive group acting on an affine variety X. Then $k[X]^G$ is finitely generated k-algebra.*

2. Reductive and linearly reductive groups

DEFINITION 14. G is linearly reductive if every finite dimensional linear representation of G decomposes as a direct sum of irreducibles.

Recall,

THEOREM 2.1 (Maschke's). *Finite groups are linearly reductive.*

PROOF OUTLINE. Suppose $G \to \mathrm{Gl}(V)$ is a finite dimensional representation of a finite group G, and $W \subset V$ is an invariant subspace. It is enough to show that W has a G-invariant complement. Choose some complement to W in V (not necessarily G-invariant) and let $p : V \to W$ be the projection along W''. Now average p over G to the G-equivariant mapping

$$p_{av}(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} p g(v).$$

Check that $p_{av}|_W = \mathrm{id}|_W$. The conclusion is the $W' := \ker p_{av}$ is a G-invariant complement of W. □

THEOREM 2.2. *In characteristic zero, a linear algebraic group G is linearly reductive iff it is reductive.*

THEOREM 2.3 (Nagata). *If G is a linearly reductive group action on an affine variety X. Then $k[X]^G$ is a finitely generated k-algebra.*

Notions of reductivity:

(1) Reductive: the unipotent radical is trivial.

(2) Linear reductive: every finite dimensional subrepresentation is completely reducible.

(3) Geometrically reductive (due Nagata): For any finite dimensional representation $G \to \mathrm{Gl}(V)$ and $0 \neq v \in V^G$ there is a homogeneous polynomial $F \in k[V]^G$ such that $F(v) \neq 0$ and $F(0) = 0$.

(4) For any G-action on an affine variety X, $k[X]^G$ is a finitely generated k-algebra.

THEOREM 2.4. *In characteristic zero, all of the above four notions are equivalent. In characteristic p, (1), (3) and (4) are equivalent.*

REMARK. (2) $\Rightarrow$(3) is easy: span(v) has a complement $W \subset V$. Let $F : V \to \mathrm{spac}(v) = \mathbb{A}^1$ be the projection along W. Then F is a G-invariant linear form and $F(v) = v \neq 0$.

Our goal is to show how (2) implies (4).

2.1. Reynolds operators. This is a way of averaging over G. Let $G \to \mathrm{Gl}(V)$ be a finite dimensional representation. Write $V = V^G \oplus W$ and $P_V : V \to V^G$ be the projection to V^G along W. Note that here W and P_V are unique. In fact W is the direct sum of the non-trivial irreducible sub-representations of V.

LEMMA 3. *Suppose V is a finite dimensional G-module and V_1 is a G-submodule. Then $P_V(x) = P_{V_1}(x)$ for all $x \in V$.*

PROOF. Write $V = V_1 \oplus V_2$ and $V_1 = V_1^G \oplus W_1$ and $V_2 = V_2^G \oplus W_2$ as above. And $V^G = V_1^G \oplus V_2^G$. If $x \in V$ then we can write is as $x = x_0 + x_1 + x_2 \in V^G \oplus W_1 \oplus W_2$. So $P_V(x) = x_0$. If $x \in V_1$ then $x_0 \in V_1^G$ and $x_2 = 0$ and we still have $P_{V_1}(x) = x_0$. □

A G-module M is called rational or locally finite dimensional if a linear action of G on vector space M, where any finite collection of elements $m_1, \cdots, m_N \in M$ lies in a finite-dimensional G-invariant subspace V of M. For instance, if G acts on an affine variety X then $M = k[X]$ is a rational G-module.

COROLLARY 3. *Let M be a rational G-module. Then P_M is well-defined.*

Proof. To define $P_M(m)$ for $m \in M$, place m into a finite dimensional G-invariant $V \subseteq M$ and define $P_M(m) := P_V(m)$. If V_1 and V_2 are both finite dimensional G-invariant subspaces of M containing m. Then there is a finite dimensional G-invariant subspace $V \subseteq M$ containing both V_1 and V_2 and then apply the lemma. □

Proposition 5. *Suppose $h : M_1 \to M_2$ is a (G-equivariant) homomorphism of rational G-modules. Then $h \circ P_{M_1} = P_{M_2} \circ h$.*

Proof. By rationality we may assume that $m \in M_1$ where M_1 is finite dimensional and $M_2 = h(M_1)$. Write $M_1 = M_1^G \oplus W_1 \oplus \cdots \oplus W_r$ where each W_i is a non-trivial irreducible representation of G. Now we apply h. By Schur's lemma $h(W_i) \cong W_i$ or is (0). Thus $M_2^G = h(M_1^G)$. Moreover, any $m \in M_1$ can be written as $m = m_0 + w_1 + \cdots + w_r \in M_1^G \oplus W_1 \oplus \cdots \oplus W_r$. Then $h(m) = h(m_0) + h(w_1) + \cdots + h(w_r)$ where $h(m_0) \in M_2^G$ and $h(w_i) \in h(W_i)$ for all i. Thus

$$P_{M_2} \circ h(m) = h(m_0) = h \circ P_{M_1}(m).$$

□

Corollary 4. *If R is a rational G-algebra and $r_0 \in R^G$. Then $P_R(r_0 r) = r_0 P_R(r)$ for any $r \in R$.*

Proof. Use the proposition with $M_1 = M_2 = R$ and h being the multiplication by r_0. □

Corollary 5. *Let $J \subset R^G$ be an ideal. Then $JR \cap R^G = J$.*

Proof. $\supseteq$ is obvious. For $\subseteq$ suppose $x = j_1 r_2 + \cdots j_n r_n \in R^G$ where $j_1, \cdots, j_n \in J$ and $r_1, \cdots, r_n \in R$. Apply P_R and get

$$x = P_R(x) = P_R(j_1 r_1) + \cdots + P_R(j_n r_n) = j_1 P_R(r_1) + \cdots + j_n P_R(r_n)$$

which is in J. □

Corollary 6. *Assume R is a rational G-algebra. If R is noetherian then R^G is noetherian.*

Proof. Let $J_1 \subset J_2 \subset \cdots \subset J_n \subset \cdots$ be an ascending chain of ideals in R^G. Then $J_1 R \subset J_2 R \subset \cdots \subset J_n R \subset \cdots$ is an ascending chain of ideals in R. Since R is noetherian $J_n R = J_{n+1} R = \cdots$ for some $n \geq 1$. Now $J_n R \cap R^G = J_{n+1} R \cap R^G = \cdots$ and apply corollary 5. □

Lemma 4. *Suppose $S = \oplus_{i=0}^{\infty} S_i$ is a graded k-algebra (satisfying $S_i S_j \subseteq S_{i+j}$). Let $M = \oplus_{i\geq 1}^{\infty} S_i$ and $x_1, \cdots, x_n$ be homogeneous elements of S of degree ≥ 1 (i.e. each x_i lies in some $S_j, j \geq 1$). Then the following are equivalent:*

a) *$S = S_0[x_1, \cdots, x_n]$ i.e. S is generated by $x_1, \cdots, x_n$ as an S_0-algebra.*

b) *$M = \sum_{i=1}^{n} S x_i$, i.e. M is generated by $x_1, \cdots, x_n$ as an ideal.*

c) *$M/M^2 = \sum_{i=1}^n S_0\overline{x_i}$, where $\overline{x_i} = x_i \mod M^2$, i.e. M/M^2 is spanned by $\overline{x_1}, \cdots, \overline{x_n}$ as an S_0-module.*

PROOF. (a) $\Rightarrow$(b): Suppose $x \in M$, and by (a) $x = \sum_{d_1,\cdots,d_n \geq 0} s_{d_1,\cdots,d_n} x_1^{d_1} \cdots x_n^{d_n}$ with $s_{d_1,\cdots,d_n} \in S_0$ for all indices. Reducing both sides mod M we see that $s_{0,\cdots,0} = 0$ and (b) follows.

(b) $\Rightarrow$(c) is obvious: Every $x \in M$ can be written as $x = \sum_{i=1}^n s_i x_i$ for some $s_i \in S$. Modulo M^2 we can replace each s_i on the right hand side by $s_i[0] \in S_0$ where $s_i = s_i[0] + s_i[1] + \cdots, s_i[k] \in S_k$. Thus in M/M^2, $\overline{x} \in \sum_{i=1}^n S_0\overline{x_i}$.

(c) $\Rightarrow$(a): Since $M = \sum_{i=1}^n S_0 x_i + M^2$ we have $M_2 = \sum_{i,j=1}^n S_0 x_i x_j + M^3$. Arguing inductively, we see that

$$M_d = \sum_{d_1+\cdots+d_n=d} S_0 x_1^{d_1} \cdots x_n^{d_n} + M^{d+1} \quad \forall d \geq 0$$

and thus

$$S = \sum_{d_1,\cdots,d_n \leq d} S_0 x_1^{d_1} \cdots x_n^{d_n} + M^{d+1} \quad \forall d \geq 0.$$

Now say $S' = S_0[x_1, \cdots, x_n]$ and observe that $S' = S$: clearly $S' + M^{d+1} = S$ for all $d \geq 0$. But every $a \in S_i$ lies in S' for any $i \geq 0$. Indeed take $d = i$ then there is $b \in S'$ such that $a = b +$ element of M^{i+1}. Thus $a = b[i] \in S'$ and this completes the proof. $\square$

PROOF OF (2) $\Rightarrow$ (4) IN THEOREM 2.4. Case 1: $X = V$ is a vector space with linear G-action. Then $R = k[X] = \oplus R_i$ where R_i is the set of homogeneous polynomials of degree d on V. Then $S = R^G = \oplus_{i \geq 0} R_i^G$ is also a graded k-algebra. Indeed, if $f = f_0 + \cdots + f_d$ lies in S, where $f_i \in R_i$ then,

$$\begin{aligned} f &= P_R(f) = P_R(f_0) + \cdots + P_R(f_d) \\ &= \underbrace{P_{R_0}(f_0)}_{\in R_0} + \cdots + \underbrace{P_{R_d}(f_d)}_{\in R_d}. \end{aligned}$$

Since $R = \oplus_{i \geq 0} R_i$ we conclude that $f_i = P_{R_i}(f_i) \in S$ as desired.

We know that R is noetherian by Hilbert's basis theorem. Hence by corollary 6 S is noetherian. Thus $M = \oplus_{i \geq 0} R_i^G$ is a finitely generated ideal. By (b) $\Rightarrow$(a) of previous lemma S is a finitely generated k-algebra.

Case 2: In the general affine case, let V be a finite dimensional G-invariant subspace of $k[X]$ containing a set of algebra generators of $k[X]$. Let $f_1, \cdots, f_n$ be a basis of V. Define

$$\begin{aligned} F : X &\to \mathbb{A}^n \\ x &\mapsto (f_1(x), \cdots, f_n(x)). \end{aligned}$$

Note that the G-action on V makes F into a G-equivariant map. Thus there is a surjective G-equivariant morphism

$$\begin{aligned} h : \overbrace{k[x_1, \cdots, x_n]}^{R} &\to k[X] \\ x_i &\mapsto f_i \end{aligned}$$

Now $k[X]^G = P_{k[X]}(k[X]) = P_{k[X]} \circ h(R) = h \circ P_R(R) = h(R^G)$. By Case 1, R^G is finitely generated, and therefore so is $k[X]^G$. $\square$

In the situation of the above theorem, we will refer to the morphism $\pi : X \to Q$ as a *quotient* or *quotient map*. Here π is induced by inclusion $k[X]^G \hookrightarrow k[X]$ as observed above. We will show that π is the categorial quotient.

PROPOSITION 6.

a) *π is onto.*

b) *If $x_1, \cdots, x_n$ are closed and G-invariant subvarieties of X then $\pi(X_1 \cap \cdots \cap X_n) = \pi(X_1) \cap \cdots \cap \pi(X_n)$.*

c) *If $Y \subset X$ is a closed and G-invariant then $\pi|_Y : Y \to \pi(Y)$ is a quotient for G-action on Y.*

d) *If $U \subset Q$ is an open subset $U \neq \varnothing$, then $k[\pi^{-1}(U)]^G = k[U]$.*

PROOF. We translate each part into algebraic language:

a') $R = k[X]$ and $S = k[X]^G = k[Q]$, point in Q corresponding to maximal ideal $J \subset S$. Want to show $JR \neq R$. But $JR \cap S = J$ by lemma 4 implying the assertion.

b') If $I_1, \cdots, I_n \subset R$ are G-invariant ideals then

$$\left(\sum_{m=1}^{n} I_m\right) \cap S = \sum_{m=1}^{n} (I_m \cap S).$$

$\supseteq$ is clear. For the converse start with $x = x_1 + \cdots + x_n \in S$, $x_m \in I_m$. But

$$x = P_R(x) = \sum_m P_R(x_m) = \sum_m \underbrace{P_{I_m}(x_m)}_{\in I_m^G = I_m \cap S} \in \sum_{m=1}^{n} (I_m \cap S).$$

c') For $I \subset R$ a G-invariant ideal, $(R/I)^G = S/I \cap S$. But consider the G-equivariant homomorphism $R \to R/I$. Then

$$(R/I)^G = P_{R/I}(R/I) = P_{R/I}(h(R)) = hP_R(R) = h(S) = S/I \cap S.$$

d') For any $f \in S$, we have that $R[1/f]^G = S[1/f]$. But this is obvious since $\frac{a}{f} \in R[\frac{1}{f}]$ is G-invariant if and only if a is G-invariant.

□

COROLLARY 7. *Each fiber of π contains exactly one closed G-orbit.*

PROOF. Suppose $q \in Q$ and $\pi^{-1}(q)$ contains closed G-orbits O_1 andO_2. Then $O_1 \cap O_2 = \varnothing$ is impossible. Because $\pi(O_1) \cap \pi(O_2)$ contains q. It remains to show that $\pi^{-1}(q)$ contains a closed G-orbit. Let $x \in \pi^{-1}(q)$ by part (a) of proposition 6 above. If Gx is closed, we are done. Otherwise take $x_1 \in \overline{Gx} \smallsetminus Gx$. Then $\overline{Gx_1} \subsetneq \overline{Gx}$ by Chevalley's theorem: Gx contains an open subset of $\overline{Gx}$. If Gx_1 is closed we are done. Otherwise, take $x_2 \in \overline{Gx_1} \smallsetminus Gx_1$ and continue. Get a desceding chain of inclusions .

$$\cdots \subsetneq \overline{Gx_1} \subsetneq \overline{Gx_1} \subsetneq \overline{Gx}.$$

By noetherian property, get a contradiciton. □

THEOREM 2.5. *Let $G, X, \pi : X \to Q$ be as in the proposition. Then π is the categorical quotient for G-action on X.*

PROOF. We need to show that the dotted arrow ψ is a morphism in the diagram

$$\begin{tikzcd} X \arrow[d, "\pi"] \arrow[dr, "\varphi"] & \\ Q \arrow[r, dashed, "\psi"] & Y \end{tikzcd}$$

Here ψ maps $q \in Q$ to $\varphi(x)$ for an arbitrary choice of $x \in \pi^{-1}(q)$ having closed orbit $G.x \subseteq X$. ψ is well-defined on points. If Y is affine it is clear that ψ is a morphism: for any $f \in k[Y]$, $\varphi^*(f)$ is a G-invariant function on X hence in $k[X]^G$ and ψ is just dual to the inclusion $\varphi^* : k[Y] \hookrightarrow k[X]^G \hookrightarrow k[X]$.

In general, we shall show that ψ is a morphism locally in a neighborhood of some $q \in Q$. Let Y_0 be an affine open in Y containing $\psi(q) = \varphi(x)$. set $Z = X \smallsetminus \varphi^{-1}(Y_0)$. By proposition 6 (b), the is $f \in k[X]^G$ which separates Z from Gx, i.e. $f|_Z \cong 0$ and $f(x) \neq 0$. replace X by X_f, Q by Q_f and Y by Y_0. By proposition 6 (d), $k[X_f]^G = k[Q_f]$. So we have reduced to the case where Y is affine. □

3. Stability

Let G be a linear reductive algebriac group and V a linear representation of G. Until now we know that $k[V]^G$ is a finitely generated k-algebra. We know that $\pi : V \to Q$, the categorical quotient map separates closed G-orbits in V. But in general, it is hard to tell which orbits are closed. We want to study conditions for this.

DEFINITION 15. $x \in V$ is called <u>unstable</u> if $x \in \pi^{-1}\pi(0)$. Equivalently, $h(x) = 0$ for any homogeneous invariant polynomial $h \in k[V]^G$ of degree ≥ 1. All other points $x \in V$ are called <u>semistable</u>. x is called <u>properly stable</u> if $\mathrm{Stab}_G(x)$ is finite and $G.x$ is closed in V. The loci of these subsets of V, are denote respectively by V^{unst}, V^{ss}, V^s.

LEMMA 5.

a) V^{unst} *is closed,*

b) V^{ss} *is open,*

c) V^s *is open.*

PROOF. (a) and (b) are obvious. For (c) let

$$\begin{aligned}\varphi : G \times V &\to V \times V \\ (g, x) &\mapsto (x, g.x)\end{aligned}$$

and Δ be the diagonal. Then the fiber of φ over (x, x) is isomorphic to the stabilizer $\mathrm{Stab}_G(x)$. By the fiber dimension theorem,

$$\{x \in V : \dim \mathrm{Stab}_G(x) \geq 1\}$$

is closed in V. Now $V^s = \pi^{-1}(Q \setminus \pi(Z))$. We know that $\pi(Z)$ is closed by proposition 6(c). Thus $Q \setminus \pi(Z)$ and consequently V^s are open. □

We know that the set of points with closed orbits is not always open. An important example is that of $G = \mathrm{Sl}_n$, acting on M_n by conjugation. Closed orbits are the orbits of diagonal matrices, and this set is not open for all $n \geq 2$.

3.1. A case study. $G = \mathbb{G}_m$ so our G-action on V is diagonalizable, i.e. in some basis of V, $T \in \mathbb{G}_m$ acts via

$$\begin{pmatrix} t^{w_1} & & 0 \\ & \ddots & \\ 0 & & t^{w_n} \end{pmatrix}$$

where $w_1, \cdots, w_n \in \mathbb{Z}$. Suppose $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is basic. Let us say that w_i is a weight associated to x if $x_i \neq 0$. Here $\mathrm{Stab}_G(x)$ being finite for the properly stable points is a redundant condition if the G-action is not trivial.

PROPOSITION 7. *Let* $W_x = \{$ *weights associated to* x $\} \subset \mathbb{Z}$. *Then*

(1) *x is properly stable if and only if W_x contains both positive and negative integers.*

(2) *x is unstable if and only if either every $w \in W_x$ is positive or every $w \in W_x$ is negative.*

LEMMA 6. *Let $h : H \to G$ be a homomorphism of algebraic groups. Then $h(H)$ is closed in G.*

PROOF. HW problem. □

Fixing $g \in \mathbb{G}_m$, let $\varphi : \mathbb{G}_m \to V$ via $g \mapsto g.v$ be given. Embed V into projective space $\mathbb{P}(V \times k)$ via $(a_1, \cdots, a_n) \mapsto (a_1 : \cdots : a_n : 1)$. Then φ extends uniquely to a morphism $\widetilde{\varphi} : \mathbb{P}^1 \to \mathbb{P}(V \times k)$ by the valuative criterion.

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[r, "\widetilde{\varphi}"] & \mathbb{P}(V \times k) \\
\mathbb{G}_m \arrow[u, hook] \arrow[r] & V \arrow[u, hook]
\end{tikzcd}$$

Explicitly speaking: $\varphi : t \mapsto (t^{w_1}x_1 : \cdots : t^{w_n}x_n : 1)$ or $t \mapsto (t^{w_1-m}x_1 : \cdots : t^{w_n-m}x_n : t^{-m})$ where $m = \min\{0\} \cup \{w_i : x_i \neq 0\}$. Then φ extends to $\mathbb{A}^1 = \mathbb{G}_m \cup \{0\}$ and similarly to the other $\mathbb{A}^1 = \mathbb{G}_m \cup \{\infty\}$. If $\widetilde{\varphi}(0)$ is in V we define $\lim_{t\to 0} \varphi(t) := \widetilde{\varphi}(0)$ otherwise we say that this limit does not exist. Similarly for $\lim_{t\to\infty} \varphi(t)$.

Example 3.1. Consider the action of $\mathbb{G}_m$ on k^5 via

$$t \mapsto \begin{pmatrix} t^{-2} & 0 & 0 & 0 & 0 \\ 0 & t^{-2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & t^2 \end{pmatrix}.$$

Then when lifting the action to $\mathbb{P}^5$ we find that

$$\lim_{t\to 0} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = (1 : 1 : 0 : 0 : 0 : 0) \in \mathbb{P}^5 \smallsetminus V$$

so the limit does not exist for our original action. However

$$\lim_{t\to 0} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = (0 : 0 : 1 : 0 : 0 : 1) \in \mathbb{P}^5$$

so the limit is $(0, 0, 1, 0, 0) \in k^5$. ⌟

Example 3.2. Consider $t \mapsto \begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}$. Then the unstable and not-properly stable points are the same: $V(x_1, x_2) \cup V(x_3)$. For $t \mapsto \begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}$,

$$\begin{aligned} V^s &= V \smallsetminus V(x_1) \cup V(x_3) \\ V^{uns} &= V(x_1, x_2) \cup V(x_2, x_3) \end{aligned}$$

⌟

Proof of proposition 7. We embed $V \subset \mathbb{P}^n$ and assume all $x_i \neq 0$ (otherwise restrict V to a subspace). Also assume $m = w_1 \leq w_2 \leq \cdots \leq w_n = M$ and $w_1 = \cdots = w_r < w_{r+1}$ and $w_s < w_{s+1} = w_{s+2} = \cdots = w_n$. Then in $\mathbb{P}^n$,

$$p = \lim_{t \to 0} t.x = \begin{cases} (x_1 : \cdots : x_r : 0 : \cdots : 0) & m < 0 \\ (x_1 : \cdots : x_r : 0 : \cdots : 0 : 1) & m = 0 \\ (0 : \cdots : 0 : 1) & m > 0 \end{cases},$$

and similarly

$$q = \lim_{t \to \infty} t.x = \begin{cases} (0 : \cdots : 0 : x_{s+1} : \cdots : x_n : 0) & M > 0 \\ (0 : \cdots : 0 : x_{s+1} : \cdots : x_n : 1) & M = 0 \\ (0 : \cdots : 0 : 1) & M < 0 \end{cases}.$$

Let X be the closure of $\mathbb{G}_m.x$ in $\mathbb{P}^n$. Then $X = \mathbb{G}_m.x \cup \{p, q\}$. If $p, q \notin V$ then $\mathbb{G}_m.x = V \cap X$ is closed in V. This happens if $m < 0$ and $M > 0$ proving sufficiency in (1). Note also that in this case $\mathrm{Stab}_{\mathbb{G}_m} x$ is finite. Conversely if x is properly stable then $\mathrm{Stab}_{\mathbb{G}_m}(x)$ is finite. Hence $(m, M) \neq (0, 0)$. Moreover $\mathbb{G}_m x$ is closed in V then $p, q \notin V$ and thus $m < 0$ and $M > 0$.

For (2) is x in unstable iff $0 \in \overline{\mathbb{G}_m x}$ iff p or q are $(0 : \cdots : 0 : 1)$ iff $m > 0$ or $M < 0$. □

Our main tool for determining stability of points on an affine space is the Hilbert-Mumford criterion. Since this is a crucial theorem in geometric invariant theory we will devote a separate chapter to the statements and proof of this theorem.

CHAPTER 5

Hilbert-Mumford Criterion

Hilbert-Mumford Criterion 0.1. Let G be a linearly reductive group acting on a vector space V. Then

(1) $x \in V$ is properly stable if and only if x is properly stable for every 1-parameter subgroup $\rho : \mathbb{G}_m \hookrightarrow G$. Equivalently $\lim_{t \to 0} \rho(t).x$ does not exist for any ρ.

(2) $x \in V$ is semistable if and only if it is semistable for any 1-parameter subgroup $\rho : \mathbb{G}_m \hookrightarrow G$. Equivalently $\lim_{t \to 0} \rho(t).x \neq 0$ for all ρ.

Note 0.3. For $\lambda : \mathbb{G}_m \to G$ if it is not injective, but $\lambda \neq 1$ then λ factors through $\lambda : \mathbb{G}_m \to \mathbb{G}_m = \mathbb{G}_m / \ker(\lambda) \xrightarrow{\lambda'} G$. If $\ker(\lambda) = \mu_d$ then the weights of $\lambda = d$ (weights of λ'). Thus can always replace λ by λ'. Hence we can always assume that λ is injective. ⌟

Example 0.4. Let $V = k^{n+1}$ be the space of binary forms of degree n,

$$f(x,y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n$$

with the natural action of $G = \mathrm{Sl}_2$ by coordinate changes. Up to conjugation, every 1-parameter subgroup has the form $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. So the weights of the action on k^{n+1} are

$$t \mapsto \begin{pmatrix} t^n & 0 & \cdots & 0 \\ 0 & t^{n-2} & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & t^{-n} \end{pmatrix}.$$

The unstables are those with coefficient a_j of $x^i y^j$ being zero where; $f(x,y)$ is a multiple of $x^{\lceil n/2 \rceil}$. Not properly stable if $a_j = 0$ whenever $j > i$; $f(x,y)$ is a multiple of $x^{\lfloor n/2 \rfloor}$. For $n = 4$ we get M_1 the coarse moduli space of curves of genus one. For $n = 6$ we get M_2 the coarse moduli space of curves of genus two. ⌟

Example 0.5. Let $V = M_n$ and Gl_n is acting by conjugation. We know that for $A \in M_n$, $G.A$ is closed in M_n if and only if A is diagonalizable. We also know that $k[V]^{\mathrm{Gl}_n}$ is generated by $e_1(A), \cdots, e_n(A)$, the coefficients of the characteristic polynomial of A. A is unstable iff $e_i(A) = 0$ for all i iff A has all eigenvalues zero, iff A is nilpotent. If A is properly stable then A is diagonalizable. What do we know about $\mathrm{Stab}_{\mathrm{GL}_n}(A)$? All polynomials in A are in it. In particular it will be infinite. (Check that the stabilizer is also infinite for the action of Sl_n as well.) So $V^s = \varnothing$. In any case we know the coarse moduli space for this action so we do not need the Hilbert-Mumford criteria for study of the orbit space here. ⌟

Example 0.6. Let $G = \mathrm{Sl}_n$ and $V = M_n \times M_n$. One can think of the elements of V as representations of the free algebra $k\{x, y\} \to M_n$. The action of any one-parameter subgroup diagonalizes as

$$\lambda : t \mapsto \mathrm{diag}(t^{\alpha_1}, \cdots, t^{\alpha_n}),$$
$$\sum_i \alpha_i = 0, \alpha_1 \geq \cdots \geq \alpha_n$$
$$\alpha_1 = \cdots = \alpha_{r_1} > \alpha_{r_1+1} = \cdots = \alpha_{r_1+r_2} > \cdots$$

So $\lambda(t)(a_{ij})_{n\times n} = (t^{\alpha_i - \alpha_j} a_{ij}), \lambda(t)(b_{ij})_{n\times n} = (t^{\alpha_i - \alpha_j} b_{ij})$. If all weights associated with (A, B) are ≥ 0 then both A and B are block-trinagular, i.e. of the form

$$\begin{pmatrix} \boxed{r_1 \times r_1} & & & * \\ & \boxed{r_2 \times r_2} & & \\ & & \ddots & \\ 0 & & & \boxed{r_s \times r_s} \end{pmatrix}.$$

So (A, B) is properly stable iff they have no common invariant subspace. In fact there is a

Theorem 0.1 (Burnside). *A collection $S \subset M_n(k)$ for algebraically closed field k of $n \times n$ matrices, has a common invariant subspace if and only if S does not generate M_n as a k-algebra.*

Let $\varphi : k\{x, y\} \to M_n$ correspond to (A, B) via $x \mapsto A, y \mapsto B$. Then what we showed is that φ is properly stable iff φ is irreducible. By Burnside's theorem this is equivalent to φ being surjective.

$\lambda(t)$ has strictly positive weights associated to (A, B) iff (A, B) are strictly block-upper triangular with blocks of size r_i:

$$\begin{pmatrix} \boxed{0_{r_1 \times r_1}} & & & * \\ & \boxed{0_{r_2 \times r_2}} & & \\ & & \ddots & \\ 0 & & & \boxed{0_{r_s \times r_s}} \end{pmatrix}.$$

So (A, B) is unstable iff A and B are stricly upper trinagular in same basis. Equivalently every monomial in A, B of degree > 0 is nilpotent. ⌟

Example 0.7. Let $V = k^{10}$ be the space of degree three forms in three variables x, y and z. Let $G = \mathrm{Sl}_3$.

LEMMA 7. *If $f(x, y, z)$ cuts out a smooth cubic in $\mathbb{P}^2$ then $f(x, y, z)$ is properly stable.*

In fact if f is smooth after a change of basis we can rewrite it in

Weierstrass form	$xy^2 = (x - \alpha_1 z)(x - \alpha_2 z)(x - \alpha_3 z)$	α_i's are distinct
or Hasse form	$x^3 + y^3 + z^3 + cxyz = 0$	$c \neq -1(??)$

We need to check that if f is not properly stable then f is singular. By the Hilbert-Mumford criterion, there is a one parameter subgroup $\lambda : \mathbb{G}_m \to \mathrm{Sl}_3$ such that every weight of λ associated to f in non-negative. After we replace λ by $g\lambda g^{-1}$ for a suitable $g \in \mathrm{Sl}_3$ and f by $g.f$ we may assume

$$\lambda(t) = \begin{pmatrix} t^\alpha & 0 & 0 \\ 0 & t^\beta & 0 \\ 0 & 0 & t^\gamma \end{pmatrix}, \alpha + \beta + \gamma = 0, \alpha \geq \beta \geq \gamma.$$

In particular $\alpha > 0$ and $\gamma < 0$.

Case $\beta \geq 0$: xz^2, yz^2 and z^3 have weights $\alpha + 2\gamma, \beta + 2\gamma, 3\gamma < 0$. Thus they enter into $f(x, y, z)$ with zero coefficient. Now $Df(0:0:1) = 0$, i.e. the curve $f = 0$ is singular at $(0:0:1)$ (and passes through it).

Case $\beta < 0$: Then every nontrivial f should be a multiple of x (otherwise it will have negative weights). Thus $f(x, y, z) = xq(x, y, z)$ where q is of degree 2. Hence this curve $f = 0$ is not smooth. ⌟

THEOREM 0.2 (Hilbert-Mumford criterion for tori - Richardson). *Let $T = \mathbb{G}_m^r$ be a torus acting on $V = k^n$. If $w \in \overline{Tv}$ for some $v, w \in V$ then there exists a one parameter subgroup $\lambda : \mathbb{G}_m \to T$ such that $\lim_{t \to 0} \lambda(t)v \in Tw$.*

REMARK. The theorem as stated becomes false if T is replaced by an arbitrary linearly reductive group. It is true if we assume that Tw is closed in V (cf. an example in HW #4).

PROOF. Diagonalize the T-action on V in basis $e_1, \cdots, e_n$:

$$t \mapsto \begin{pmatrix} \chi_1(t) & & \\ & \ddots & \\ 0 & & \chi_n(t) \end{pmatrix}$$

where $\chi_1, \cdots, \chi_n : T \to \mathbb{G}_m$ are characters. Say $v = \left(x_1, \cdots, x_n\right)^t$. We may then assume $x_1, \cdots, x_n \neq 0$. Moreover after rescaling $e_1, \cdots, e_n$ we may assume $v = \left(1, \cdots, 1\right)^t$.

LEMMA 8. *There is $h \in T$ such that* $hw = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \# = s$ *for some* $0 \leq s \leq n-1$.

PROOF OF LEMMA. After permuting coordinates we may assume $w' = (a_1, \cdots, a_s, 0, \cdots, 0)^t$ where $a_1, \cdots, a_s \neq 0$. Now it is enough to show that there is $h \in T$ such that $\chi_1(h) = a_1, \cdots, \chi_s(h) = a_s$. We know that $w = \left(a_1, \cdots, a_s, 0, \cdots, 0\right)^t$ lies in the closure of $Tv = \begin{pmatrix} \chi_1(t) \\ \vdots \\ \chi_n(t) \end{pmatrix}$ in V. Thus $\left(a_1, \cdots, a_s\right)^t$ lies in closure of $\begin{pmatrix} \chi_1(t) \\ \vdots \\ \chi_s(t) \end{pmatrix}$ in k^s as t ranges over T, and this is in $\mathbb{G}_m^s$. In other words $\left(a_1, \cdots, a_s\right)^t$ is in the closure of the image of the homomorphism

$$\begin{aligned} T &\to \mathbb{G}_m^s \\ t &\mapsto \begin{pmatrix} \chi_1(t) \\ \vdots \\ \chi_s(t) \end{pmatrix}. \end{aligned}$$

We know that this image is closed (by a homework problem). Thus $\begin{pmatrix} a_1 \\ \vdots \\ a_s \end{pmatrix} = \begin{pmatrix} \chi_1(h) \\ \vdots \\ \chi_s(h) \end{pmatrix}$ for some $h \in T$ and thus

$$h^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

□

For $T = \mathbb{G}_m^r$ a character $\chi \in X(T)$ has the form $t_1^{d_1} \cdots t_r^{d_r}$ for $d_1, \cdots, d_r \in \mathbb{Z}$. We identify χ with $(d_1, \cdots, d_r)$ and $X(T)$ with $\mathbb{Z}^r$.

LEMMA 9. *Let $X(T)$ be the group of characters of T and $C = \mathrm{Span}_{\mathbb{Q}^{\geq 0}}(\chi_1, \cdots, \chi_n)$ in $X(T) \otimes \mathbb{Q} = \mathbb{Q}^r$. Then $C \cap (-C) \subset \mathrm{Span}_{\mathbb{Q}}(\chi_1, \cdots, \chi_s)$.*

PROOF. Otherwise there are $d_1, \cdots, d_n, e_1, \cdots, e_n \in \mathbb{Z}^{\geq 0}$ such that

$$\chi_1^{d_1} \cdots \chi_n^{d_n} = \chi_1^{-e_1} \cdots \chi_n^{-e_n}$$

and $d_i \neq 0$ for some $i > s$. Now $\chi_1^{d_1+e_1} \cdots \chi_n^{d_n+e_n} \cong 1$. In particular $f(x_1, \cdots, x_n) = x_1^{d_1+e_1} \cdots x_n^{d_n+e_n}$ is a T-invariant monomial on V. So since $w \in \overline{Tv}$, $f(w) = f(v)$. But $f(v) = 1$ and $f(w) = 0$. $\square$

Suppose $\chi_j(t_1, \cdots, t_r) = t_1^{a_{j1}} \cdots t_r^{a_{jr}}, 1 \geq j \geq n$. We are looking for $\lambda(t) = (t^{b_1}, \cdots, t^{b_r}) : \mathbb{G}_m \to T$, so that the action of λ on V is given via

$$t \mapsto \begin{pmatrix} t^{b_1 a_{11} + \cdots + b_r a_{1r}} & & \\ & \ddots & \\ & & t^{b_1 a_{n1} + \cdots + b_r a_{nr}} \end{pmatrix}.$$

So by the identification $X(T) \cong \mathbb{Z}^r$, say $\chi_i \mapsto p_i = (a_{i1}, \cdots, a_{ir})$. Then λ corresponds to linear form $\ell : (a_1, \cdots, a_r) \mapsto b_1 a_1 + \cdots b_r a_r$. And what we need is to pick ℓ such that

$$\ell(p_1) = \cdots = \ell(p_s) = 0, \ell(p_{s+1}), \cdots, \ell(p_n) > 0.$$

Let $C = \mathrm{Span}_{\mathbb{Q}^{\geq 0}}(p_1, \cdots, p_n)$. We mod out $\mathbb{Q}^r$ by $\mathrm{Span}_{\mathbb{Q}}(p_1, \cdots, p_s)$. Let π be the projection

$$\mathbb{Q}^r \to W = \mathbb{Q}^r / \mathrm{Span}_{\mathbb{Q}}(p_1, \cdots, p_s).$$

For $\overline{C} = \mathrm{Span}_{\mathbb{Q} \geq 0}(\overline{p}_{s+1}, \cdots, \overline{p}_n)$ lemma above implies that $\overline{C} \cap - \overline{C} = (0)$ in W. We are reduced to showing that a linear functional $\overline{\ell} : W \to \mathbb{Q}$ exists such that $\overline{\ell}(p_i) > 0$ for all $i = s+1, \cdots, n$.

Separation Theorem. Let X and Y be convex subsets of $\mathbb{R}^m$, X compact and Y closed and $X \cap Y = \varnothing$. Then there is a linear functional $\overline{\ell} : \mathbb{R}^m \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$ such that

$$\ell(y) \leq \alpha < \beta \leq \ell(x), \forall x \in X, y \in Y.$$

In out situation, let $C_{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}^{\geq 0}}(p_1, \cdots, p_n)$. Then $C_{\mathbb{R}} \cap (-C_{\mathbb{R}}) = (0)$ by a Homework 3, problem 5. Let X be the convex hull of $p_1, \cdots, p_m$,

$$X = \{c_1 p_1 + \cdots + c_m p_m : c_1, \cdots, c_m \geq 0, c_1 + \cdots + c_m = 0\}.$$

From the construction of p_i's we have that

$$\alpha_1 p_1 + \cdots + \alpha_m p_m = -(\beta_1 p_1 + \cdots + \beta_m p_m), \alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_m \geq 0$$

is only possible if $\alpha = \cdots = \alpha_m = \beta_1 = \cdots = \beta_m = 0$. This shows that the assumption of the Separation Theorem

$$X \cap (-C_{\mathbb{R}}) = (0)$$

holds, so there exist $\alpha_1, \cdots, \alpha_n \in \mathbb{R}$ and $\alpha < \beta$ such that

$$\ell_a = a_1 x_1 + \cdots + a_n x_n, \quad \text{satisfies } \ell_a(y) \leq \alpha < \beta < \ell_a(x), \forall y \in -C_{\mathbb{R}}, x \in X.$$

Since $0 \in -C_{\mathbb{R}}$, $0 = \ell_a(y) \geq \alpha$. Thus $\beta > 0$. Now, since $p_1, \cdots, p_m \in X$ we have

$$\ell_a(o_1), \cdots, \ell_a(p_m) \geq \beta > 0.$$

By continuity, $\ell_b(p_1), \cdots, \ell_b(p_m) > 0$ for any b sufficiently close to a. Now choose $b = (b_1, \cdots, b_n) \in \mathbb{Q}^n$ and multiply b by a common denominator of $b_1, \cdots, b_n$ to get a linear form $\ell_{Nb} = N\ell_b(x_1, \cdots, x_n)$ such that

$$\ell_{Nb}(p_i) > 0 \forall i, \text{ and } Nb \in \mathbb{Z}^a.$$

□

Now we prove the Hilbert-Mumford criteria in the general form we restate here. Unfortunately the proof only works over the complex numbers, $k = \mathbb{C}$.

THEOREM 0.3. *Let $G \to \mathrm{Gl}(V)$ be a linearly reductive group with a linear representation $G \to \mathrm{Gl}(V)$. For $x \in V$ let $Y \subset V$ be G-invariant and closed subset such that $Y \cap \overline{G.x} \neq \varnothing$. Then there is a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t).x \in Y$.*

PROOF DUE TO RICHARDSON. By Hilbert-Mumford criteria for tori we may assume $Y \cap \overline{Sx} = \varnothing$ for every maximal torus S of G. We will show that $\overline{Gx} \cap Y =$ in that case.

What we need is a *singular value decomposition*, $G = MTM$ where M is compact and T is a maximal torus. For example for $G = \mathrm{Gl}_n$ we can put $M = \mathrm{U}_n$ and T the group of diagonal non-singular matrices. For all $m \in M$, $\overline{(m^{-1}Tm)x} \cap Y = \varnothing$. Thus $m^{-1}\overline{(Tmx)} \cap Y = \varnothing$ implying

$$\overline{T(mx)} \cap Y = \varnothing.$$

$\overline{T.mx}$ and Y are both closed and T-invariant in V. By proposition 6 there is $f_m \in k[V]^T$ such that

$$f_m(mx) = 1 \text{ and } f_m \cong 0 \text{ on } Y.$$

Now let $U_m = \{v \in V : |f_m(v)| \neq 0\}$. Then $M.x$ is covered by opens $\{U_m : m \in M\}$. Since Mx is compact there is a finite subcover $U_{m_1}, \cdots, U_{m_N}$. Now consider the continuous function

$$f(v) = |f_{m_1}(v)| + \cdots + |f_{m_N}(v)|$$

This is a continuous T-invariant non-negative function $V \to \mathbb{R}$, $f \cong 0$ on Y and $|f| > 1/2$ on $T.Mx$. Thus on $\overline{TMx}$, we have $|f| \geq 1/2$. This shows that $\overline{TMx} \cap Y = \varnothing$ where the closure is this time taken in the analytic topology.

We will be done if we show that $M.\overline{TMx}$ is closed in V; then

$$MTMx \subset M\overline{TMx} = \overline{MTMx}$$

implying

$$\overline{Gx} \cap Y = \overline{MTMx} \cap Y = M\overline{MTx} \cap Y = M(\overline{TMx} \cap Y) = \varnothing.$$

This follows from the

LEMMA 10. *Let $M \to \mathrm{Gl}_n(\mathbb{R})$ be a representation of a compact Lie group M. If $Z \subset \mathbb{R}^n$ is closed then MZ is closed.*

PROOF. Suppose $y \in \overline{MZ}$. Want to show $y \in MZ$. We need to show $y \in MZ$. Let B be the closed unit ball centered at Y. Then

$$y \in \overline{MZ \cap B} \subset \overline{M(Z \cap MB)}.$$

Since $Z \cap MB$ is compact, $M.(Z \cap MB)$ is compact and therefore closed. Thus

$$y \in \overline{M(Z \cap MB)} = M.(Z \cap MB) \subset M.Z.$$

□

Note finally that since $G.x$ is open in the Zariski topology the analytic closure of it is the same as the Zariski closure. This completes the proof. □

Remark. The nature of the proof of such decomposition is analytic. This is what restricts us to the complex numbers. Likewise, singular value decomposition works for Sl_n or other proper subgroups of Gl_n but for other groups we may not necessarily have such a decomposition.

Remark. We will call theorem 0.3, Richardson's theorem for further reference. The following shows how this theorem implies the Hilbert-Mumford criteria in the complex case. Before that we state a non-trivial result that we will be using in the proof.

Theorem 0.4 (Matsushima). *If G is a linearly reductive group over complex numbers, $\mathrm{Stab}_G(x)$ is reductive if and only if Gx is affine.*

Proof of Hilbert-Mumford criteria in the complex case. Suppose $x \in V$ is unstable, i.e. $\overline{Gx} \in 0$. Then the existence of λ follows from Richardson's theorem with $Y = \{0\}$. The converse is obvious.

For the second part assume x is not properly stable. Then either Gx is not closed, or $\mathrm{Stab}_G(x)$ is infinite. In the former case let Y be the unique closed G-orbit in $\overline{Gx}$ and apply Richardson's theorem to this Y. Say Gx is closed but the stabilizer is infinite. If we know that $\mathrm{Stab}_G(x)$ is reductive then there is a one-dimensional torus in it so we can take λ such that $\lambda(\mathbb{G}_m) \subset \mathrm{Stab}_G(x)$. This non-trivial fact follows from 0.4.

Conversely, assume x is properly stable. We want to show that $\lim_{t\to 0} \lambda(t)x$ does not exist for any $\lambda : \mathbb{G}_m \to G$. Otherwise, say $x_0 = \lim_{t\to 0} \lambda(t).x$. $x_0 \in \overline{Gx} = Gx$. Since x is properly stable, x_0 is properly stable. But $\mathrm{Stab}_G(x_0) \supset \lambda(\mathbb{G}_m)$ hence $\mathrm{Stab}_G(x_0)$ is infinite.

□

Remark. Mumford's proof of Hilbert-Mumford criteria goes as follows[**4**]: Hilbert finds a curve $C \subset G$ such that $\overline{C.x} \cap Y \neq \varnothing$. The question is how to get a one-parameter subgroup out of this curve. Say $C \subset G = \mathrm{GL}_n \subset M_{n\times n} \subset \mathbb{P}^{n^2}$. The resolve the singularities of C is $\mathbb{P}^{n^2}$. We get C^{smooth} which contains a point p ??. $\mathcal{O}_p(C^{smooth})$ is a local ring, and we pass to completion $\widehat{\mathcal{O}}_p(C^{smooth}) \cong k[\![t]\!]$ which is a DVR and we can write every element as ut^{α}. This gives rise to a $k((t))$-point of $G = \mathrm{GL}_n$, i.e. an invertible matrix $A = (a_{ij}(t))$ where $a_{ij}(t) \in k((t))$. Write $A = t^e(b_{ij}(t))$ where $b_{ij}(t) \in k[\![t]\!]$. The *Smith normal form* theorem, implies that there are $h_1, h_2 \in \mathrm{GL}_n(k[\![t]\!])$ such that

$$h_1(b_{ij}(t)h_2 = \begin{pmatrix} t^{d_1} & & \\ & \ddots & \\ & & t^{d_r} \end{pmatrix}; d_1 \geq d_2 \geq \cdots \geq d_r.$$

Then $h_1 A h_2 = \begin{pmatrix} t^{e_1} & & \\ & \ddots & \\ & & t^{e_r} \end{pmatrix}, e_i = d_i - e$. Now we use this to make a desired one-parameter subgroup. This is what Hilbert did. Mumford extends this by substituting the Smith normal form with analogous tools in the case of other groups.

LEMMA 11. *Let G be a linearly reductive group and $G \to \mathrm{Gl}(V)$ a linear representation. Let $X \subset V$ be a closed G-invariant irreducible, subvariety and finally $\pi : X \to X/\!\!/G$ the categorical quotient. Assume $X^{ps} \neq \varnothing$. Then*

(1) $\dim X/\!\!/G = \dim(X) - \dim G$,

(2) $k(X)^G = \mathit{ff} k[X]^G := k(X/\!\!/G)$.

PROOF. For $x \in X^{ps} \subset_{open} X$, $\pi^{-1}\pi(x) = Gx$. Every irreducible component of Gx has dimension $\dim G$, proving first claim by fiber dimension theorem. For the second assertion the nontrivial case is the inclusion $\subset$. Choose $f_1, \cdots, f_n \in k(X)^G$ as generators of the field extension $k \subset k(X)^G$. Consider the rational map $\varphi : X \dashrightarrow \mathbb{A}^n$. via $x \mapsto (f_1(x), \cdots, f_n(x))$. Let Z be the indeterminacy locus of φ. Then Z is closed and G-invariant subset of X.

Claim: $\pi(Z) \subsetneq X/\!\!/G$. Indeed, both $U = X \smallsetminus Z$ and X^{ps} are G-invariant dense open in X. If $x \in U \cap X^{ps}$ then Gx and Z are both closed G-invariant in V and $Z \cap Gx = \varnothing$. As we showed earlier, $\pi(Z)$ and $\pi(Gx)$ are closed in $X/\!\!/G$ and $\pi(Z) \cap \pi(Gx) = \varnothing$. Thus there is $0 \neq f \in k[X]^G$ such that $f \cong 0$ on Z. Now replace X by $X_f = \{x \in X : f \neq 0\}$ and $X/\!\!/G$ by $(X/\!\!/G)_f$: $\varphi : X_f \to \mathbb{A}^n$ is regular. By the universal property of the categorical quotient, φ factors through $\pi|_{X_f} : X_f \to (X/\!\!/G)_f$ via $\psi : (X/\!\!/G)_f \to \mathbb{A}^n$. In other words,

$$f_1, \cdots, f_n \in k[(X/\!\!/G)_f] = k[X/\!\!/G][\frac{1}{f}] = k[X]^G[\frac{1}{f}].$$

Therefore $f_1, \cdots, f_n \in \mathrm{ff}\, k[X]^G$. □

REMARK. We did not need finite generation of $k(X)^G$ although it is the case here.

CHAPTER 6

Categorical quotients for projective varieties

Let $X \subset \mathbb{P}(V)$ be a projective, G-invariant subvariety where $G \to \mathrm{Gl}(V)$ is a linear reductive group. We want to construct the categorical quotient in this situation. It will be $\mathcal{P}roj\, k[X^{aff}]^G$.

Recall that given a graded k-algebra R, with $R_0 = k$ an affine open $(\mathcal{P}roj\, R)_F \cong_\varphi \mathcal{S}pec(R_f)_0$ via $\mathfrak{p} \mapsto R_f \cap (R_f)_0$ for all homogeneous prime ideals $\mathfrak{p} \subset R$, such that $\oplus_{i\geq 1} R_i \not\subset$. The inverse mapping is just the contraction $\mathfrak{q} \mapsto \mathfrak{q} \cap R$ for any prime $\mathfrak{q} \in \mathcal{S}pec(R_f)_0$,

$$\mathfrak{q} \subset k[\frac{g}{f^n} : f \in R_m, m = \deg(f).n, n \in \mathbb{Z}_{\geq 0}].$$

Definition 16. Let G be a linear reductive group and $G \to \mathrm{Gl}(V)$ a given finite dimensional represenation. $X \subset \mathbb{P}(V)$ be a G-invariant closed irreducible projective variety. Let

$$X^{ss} /\!\!/ G := Q = \mathcal{P}roj\, k[X^{aff}]^G,$$

where $X^{aff} \subset V$ is the affine cone over X.

The inclusion $k[X^{aff}]^G \hookrightarrow k[X^{aff}]$ induces a rational map

$$\pi : X \dashrightarrow Q.$$

In concrete terms, if $f_0, \cdots, f_N$ are k-algebra generators of $k[V]^G$ such that f_i is homogeneous of degree d_i then

$$\begin{aligned} \pi : X &\dashrightarrow \mathbb{P}^N_{d_0,\cdots,d_N} \\ x &\mapsto [f_0(x) : \cdots : f_N(x)]. \end{aligned}$$

The unstable, semistable and properly stable loci descend from X^{aff} to X. In fact the following are equivalent

(1) π is not defined at $x \in X$

(2) $f_0(x) = \cdots = f_N(x) = 0$

(3) $f(x) = 0$ for all homogeneous $f \in k[X^{aff}]^G$ of degree ≥ 1.

(4) $f(x) = f(0)$ for any $f \in k[X^{aff}]^G$,

(5) $x^{aff} \in X^{aff}$ being unstable, and

(6) $x \in X^{unst}$.

LEMMA 12. *π is regular on X^{ss}.*

Let $R = k[V]^G = \oplus_{i=0}^{\infty} R_i$ where $R_i = k[V]^G$ is the space of degree i, G-invariant polynomials on V. For $f \in R_d (d \geq 1)$, let $U_f = (\mathcal{P}roj\, R)_f$ as above. This is an affine subvariety of $Q = \mathcal{P}roj\, R$. Now let

$$X_f = \{x \in X : f(x) \neq 0\}$$

be a G-invariant open affine subvariety of X.

$$\begin{tikzcd}
 & & (X^{aff})^{ss} \arrow[d] & \\
X_f \arrow[d] & \subset & X^{ss} \arrow[r] \arrow[d] & Y \\
U_f & \subset & X^{ss}/\!\!/G \arrow[ur, dashed] &
\end{tikzcd}$$

LEMMA 13. *Let $x \in X_f \subset X \subset \mathbb{P}(V)$ and $x^{\mathrm{aff}} \in (X^{\mathrm{aff}})_f \subset X^{\mathrm{aff}} \subset V$. Then*

(1) x is unstable (respectively properly stably), in X_f if and only if x^{aff} is unstable (respectively properly stable) in X^{aff}.

(2) The categorical quotient for the G-action on X_f and π coincide on X_f^{ss}.

PROPOSITION 8. *$\pi : X^{ss} \to X^{ss}/\!\!/G$ is the categorical quotient ofr the G-action on X^{ss}.*

PROOF. Clearly π is constant on orbits. Now suppose $\varphi : X^{ss} \to Y$ is constant on orbits. We need to show that φ factor through

$$\begin{tikzcd}
X^{ss} \arrow[d] \arrow[dr, "\varphi"] & \\
X^{ss}/\!\!/G \arrow[r, dashed] & Y
\end{tikzcd} .$$

Cover $X^{ss}/\!\!/G$ by affine opens U_f as above. Over each U_f we have

$$\begin{tikzcd}
X_f^{ss} \arrow[d, "\pi"'] \arrow[dr, "\varphi"] & \\
U_f \arrow[r, dashed, "\psi"'] & Y
\end{tikzcd} ,$$

where π is the categorical quotient. Thus ψ_f exists and is unique. By uniqueness, $\psi_f : U_f \to Y$ patch together to give

$$\psi : X^{ss}/\!\!/G \to Y.$$

□

Let G be a linearly reductive group, $G \to \mathrm{Gl}(V)$ a linear representation and $\dim V = n$. Given a homogeneous invariant $f \in k[V]^G$ of degree $d \geq 1$, set

$$(k[V]_f)_0 = k[\frac{M}{f} : \ M \text{ is a monomial in } x_1, \cdots, x_n \text{ of degree } d \ ].$$

Here $x_1, \cdots, x_n$ is a basis of V^*. Embed

$$\mathbb{P}(V)_f := \mathcal{S}pec(k[V]_f)_0 = \{x \in \mathbb{P}(V) : f(x) \neq 0\}$$

into $\mathbb{A}^N$ using above generators. Here $N = \binom{n+d-1}{n-1}$ is the number of monomials of degree d in $x_1, \cdots, x_n$.

Lemma 14. *Suppose $x \in \mathbb{P}(V)$, $f(x) \neq 0$ and $x^{\mathrm{aff}} \in V$ is a representative of x in V.*

(1) x is semi-stable in $\mathbb{A}^N$ if and only if x^{aff} is semistable in V.

(2) x is properly stable in $\mathbb{A}^N$ if and only if x^{aff} is properly stable in V.

Proof. By Hilbert-Mumford criterion, we may assume $G = \mathbb{G}_m$. Diagonalize the action of $\mathbb{G}_m$ on V in a basis $e_1, \cdots, e_n$ with associated characters $t \mapsto t^{w_1}, \cdots, t \mapsto t^{w_n}$. Without loss of generality assume $x^{aff} = e_1 + \cdots + e_m$. Denote the dual basis by $x_1, \cdots, x_n$. Weights associated to x are therefore $\{w_1, \cdots, w_m\} = W$.

In $\mathbb{A}^n$, x has coordinates

$$\left(y_{d_1, \cdots, d_n} = \begin{cases} \frac{1}{f(x^{aff})} & d_{m+1} = \cdots = d_n = 0 \\ 0 & \text{otherwise} \end{cases} : d_1, \cdots, d_n \geq 0, d_1 + \cdots + d_n = d \right).$$

Weights associated to x in $\mathbb{A}^n$ are $d_1 w_1 + \cdots, e_m w_m = w_{d_1, \cdots, d_m}$ where $d_1, \cdots, d_m \geq 0$ and $d_1 + \cdots + d_m = n$. Now observe that all $w_1, \cdots, w_m > 0$ if and only if all $w_{d_!, \cdots, d_m} > 0$ Similarly for ≥ 0, ≤ 0 and < 0. □

Remark. If $G = T$ is a torus, the Hilbert-Mumford criterion can be restated in terms of $C(x)$ the covex hull of the set of weights in $X(T) \otimes \mathbb{R}$ associated to x, $\chi_1, \cdots, \chi_m$. In fact x is unstable if and only if $0 \notin C(X)$ and is properly stable if and only if 0 is in the interior of $C(x)$.

In the above proof $T = \mathbb{G}_m, X(T) = \mathbb{Z}$.

$$W' = W + \cdots + W = W^{+d}$$

is the set of characters associated to $x \in \mathbb{A}^n$ and the cover hull of W' will be

$$d(\text{convex hyll of } W).$$

Thus $0 \in C(x)$ if and only if $0 \in C(x^{aff})$ and 0 is in the interior of $C(x)$ if and only if 0 is in the interior of $C(x^{aff})$.

THEOREM 0.5. *Let $G \to \mathrm{Gl}(V)$ be given as above. $X \subset \mathbb{P}(V)$ be a closed G-invariant irreducible subset and set*

$$X^{ss}/\!\!/G := \mathcal{P}roj\, k[X^{\mathrm{aff}}]^G.$$

Then

(1) *$\pi : X \dashrightarrow X^{ss}/\!\!/G$ is regular on X^{ss}.*

(2) *$X^{ss}/\!\!/G$ is the categorical quotient for the G-action on X^{ss}.*

(3) *If $x \in X^{ps}$ then $\pi(x) = \pi(y)$ precisely when $y \in Gx$.*

(4) *If $X^{ps} \neq \varnothing$ then*

$$\dim X^{ss}/\!\!/G = \dim X - \dim G.$$

(5) *$X^{ss}/\!\!/G$ is a projective variety.*

For the proof we need the following

LEMMA 15. *Suppose $x, y \in V^{ss}$. Then there is a homogenous $h \in k[V]^G$ of degree ≥ 1 such that $h(x) \neq 0$ and $h(y) \neq 0$.*

PROOF. Case 1. $\overline{Gx} \cap \overline{Gy} \neq \varnothing$. Say $z \in \overline{Gx} \cap \overline{Gy}$. Then

$$f(x) = f(y) = f(z), \forall f \in k[V]^G.$$

Since $x \in V^{ss}$ there is a homogeneous $h \in k[V]^G$ of degree ≥ 1, such that $h(x) \neq 0$ and we are done.

Case 2. $\overline{Gx}$, $\overline{Gy}$ and $\{0\}$ are disjoint closed G-invariant subsets of V. Hence

$$\pi(\overline{Gx}) = \pi(x), \pi(\overline{Gy}) = \pi(y), \text{ and } \pi(0)$$

are distinct. Here $\pi : V \to V/\!\!/G$ is the categorical quotient. In other words, there is $f \in k[V]^G$ such that $f(0) = 0$ but $f(x) \neq 0$ and $f(y) \neq 0$. Write $f = f_1 + \cdots + f_n$, where f_i's are homogenous of degree i. Clearly each $f_i \in k[V]^G$. Since $f(x) \neq 0$, $f_i(x) \neq 0$ for some i. Similarly $f_j(y) \neq 0$ for some j. Now consider three possibilities: if $f_i(y) \neq 0$ take $h = f_i$, if $f_j(x) \neq 0$ then take $h = f_j$. If $f_i(y) = 0$ and $f_j(x) = 0$ take $h = f_i^j + f_j^i$. ☐

PROOF. (0), (1) and (4) were proved last time. For (2) say $x, y \in X_f$ for some homogeneous $f \in k[V]^G$. Indeed let

$$Z_1 = \overline{Gx^{aff}} \cup \overline{Gy^{aff}}, Z_2 = \{0\}.$$

Since x^{aff}, y^{aff} are semistable, $Z_1 \cap Z_2 = \varnothing$. Thus there is $f \in k[V]^G$ such that

$$f(0) = 0, f(x) \neq 0, f(y) \neq 0.$$

We can choose f homogeneous by lemma 15. In X_f we know that $\pi(x) = \pi(t)$ implies $y \in Gx$ becasuse $x \in X_f^{ps}$. To compute $\dim X^{ss}/\!\!/G$ in (3) we pass to open subsets

$$X_f^{ss} \xrightarrow{\pi_f} X_f^{ss}/\!\!/G$$

since $X_f^{ps} \neq \varnothing$ know that

$$\dim X^{ss}/\!\!/G = \dim X_f/\!\!/G = \dim X_f - \dim G = \dim X - \dim G.$$

□

Remark. Look at $x \in \mathbb{P}(V)^{ss}$ mapping to $\pi(x) \in \mathbb{P}(V)^{ss}/\!\!/G$. If $x \in \mathbb{P}(V)^{ps}$ then Gx is closed in $\mathbb{P}(V)^{ss}$ but not in $\mathbb{P}(V)$. Indeed choose a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$. Diagonalize λ so that for

$$x = (x_1 : \cdots : x_n) \in \mathbb{P}(V), \lambda(t).x = (t^{w_1} x_1 : \cdots : t^{w_n} x_n).$$

We may assume $x_1, \cdots, x_m \neq 0$ and $x_{m+1} = \cdots = x_n = 0$. Since x is properly stable, $w_i > 0$, $w_m < 0$. Assume

$$w_1 \geq \cdots \geq w_m, w_1 = \cdots = w_s > w_{s+1}.$$

Then $\lim_{t\to\infty} \lambda(t).x = (x_1 : \cdots : x_s : 0 : \cdots : 0) = y \in \mathbb{P}(V)$. Note that

$$\begin{pmatrix} t^{w_1} & & \\ & \ddots & \\ & & t^{w_n} \end{pmatrix} \text{ and } \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & t^{w_{s+1}-w_1} & & \\ & & & & \ddots & \\ & & & & & t^{w_n - w_1} \end{pmatrix}$$

act in the same way on $\mathbb{P}(V)$. We have $y \in \overline{Gx}$ and in fact $y \in \mathbb{P}(V)^{unst}$, the reason is that weights of λ associated to y are $w_1, \cdots, w_s > 0$. On the other hand every point in Gx is properly stable. Thus $y \in \overline{Gx} \setminus Gx$ and this Gx is not closed in $\mathbb{P}(V)$.

CHAPTER 7

Application - Construction of the moduli of elliptic curves

1. Review on curves

We review some of the theory of curves in algebraic geometry and fix notations. So let X be a curve, i.e. a smooth projective irreducible variety of dimension one and let K be its canonical line bundle. Recall that $\mathcal{O}_p(X)$ is a discrete valuation ring

$$\nu_p : \mathcal{O}_p(X)^* \to \mathbb{N}$$

which assigns to f its order of vanishing at p. One can extend ν_p to $k(X)^* \to \mathbb{Z}$. We say that f has a zero at p or order (or multiplicity) d if $\nu_p(f) = d \geq 0$, and a pole of order d if $\nu_p(f) = -d < 0$.

Recall that divisor on X are formal finite sums $D = \sum_{p \in X} n_p[p]$. The divisor associated to $f \in k(X)^*$ is

$$\operatorname{div}(f) = \sum_{p \in X} \nu_p(f)[p].$$

We know that $\deg(\operatorname{div}(f)) = 0$ where $\deg(\sum_{p \in E} n_p[p]) = \sum_{p \in X} n_p$.

Associated to a divisor D there is a k-vector space

$$L(D) := \{f \in k(X)^* : \operatorname{div}(f) + D \geq 0\} \cup \{0\}$$

of dimension $\ell(D)$, and Riemann-Roch theorem says

$$\ell(D) - \ell(K - D) = \deg D - g + 1.$$

Immediate consequences to recall are the cases

(1) $D = 0$; since the only regular functions on X are constants, this implies the $\ell(K) = g$, the dimension of the space of regular differential forms on X.

(2) $D = K$ implies $\deg K = 2g - 2$.

Linearly equivalence: Recall that any two meromorphic differential forms differ by a rational function coefficient. In other words $K = \operatorname{div}(w)$ for meromorphic function ω is independent of the choice of this function up to linear equivalence.

Case of elliptic curves: Here $g = 1, \ell(K) = 1, \deg(K) = 0$. In fact K is linearly equivalence to an effective divisor and we can let $K = 0$. Then Riemann-Roch can be restated as

$$\ell(D) - \ell(-D) = \deg D + 1 - g = \deg D.$$

So

$$\ell(D) = \begin{cases} \deg D & \deg D > 0 \\ 0 & \deg D < 0. \end{cases}$$

Also note that $L(D) \neq (0)$ if and only if D is linearly equivalent to an effective divisor. So if $\deg D = 0$ then

$$\ell(D) = \begin{cases} 1 & \text{if } D = \operatorname{div}(f), \text{ for some } f \\ 0 & \text{otherwise.} \end{cases}$$

In particular $D + \operatorname{div}(1/f) \geq 0$ and $D + \operatorname{div}(g) \geq 0$ for any meromorphic f, g. So

$$\operatorname{div}(g) - \operatorname{div}(1/f) \geq 0$$

so $\operatorname{div}(g.f) \geq 0$ and this means $f.g$ is a constant.

Lemma 16. *Choose a point A on X. Then every divisor D of degree 0 on X can be uniquely written as $[p] - [A]$ for some (unique) point $p \in X$.*

Proof. Let D be a divisor of degree zero on X. Then $\deg(D + [A]) = 1$ and

$$\ell(D + [A]) = 1.$$

So take $0 \neq f \in L(D + [A])$. Still $\deg(D + [A] + \operatorname{div}(f)) = 1$ and if effective, hence

$$D + [A] + \operatorname{div}(f) = [p]$$

for some $p \in X$. Thus $D + \operatorname{div}(f) = [p] - [A]$, i.e. D is linearly equivalenct to $[p] - [A]$. For uniqueness if

$$[P] - [A] \sim [Q] - [A]$$

then $[P] + \operatorname{div}(f) = [Q]$ for some $f \in k(X)^*$. Thus $f \in: ([P])$. Also $1 \in L([P])$. Since $\ell([P]) = 1$, f is a scalar multiple of 1, i.e a constant. We conclude that $[Q] = [P] + \operatorname{div}(f) = [P]$. □

Let $P, Q \in X$, then

$$D = [P] + [Q] - 2[A] = [R] - [A]$$

by the above lemma. $(P, Q) \mapsto R$ defines an operation $X \times X \xrightarrow{\oplus} X$ which is a commutative group on points of X.

Theorem 1.1. *The group operations are given by morphisms, i.e. X is an algebraic group, with operations as above. A is the origin for this group structure.*

2. Elliptic curves as ramified coverings of the projective line

If $f : C_1 \to C_2$ is morphism of smooth curves of degree $d = [k(C_1) : k(C_2)]$ then recall the Hurwitz formula

$$2 - 2g(C_1) = d(2 - 2g(C_2)) - \sum_{p \in C_1} (e_p - 1)$$

where e_p is the ramification index of f at p

$$e_p = \mathrm{ord}_p(f^* z)$$

for a choice of uniformizing parameter z on C_2 at $f(p)$.

Now let E be an elliptic curve $A \in E$. $L(A)$ is spanned by 1 as a k-vector space. Choose a basis $\{1, f\}$ of $L(2A) \supset L(A)$ with

$$f : E \dashrightarrow \mathbb{A}^1.$$

If we complete f to a regular map $f : E \to \mathbb{P}^1$ we get $f^{-1}(\infty) = A$. If $\lambda \in \mathbb{A}^1$ then the divisor of poles of $f - \lambda$ is $2[A]$ so the divisor of zeroes of $f - \lambda$ has degree 2. Thus

$$f : E \to \mathbb{P}^1$$

is a $2:1$ covering. By Hurwitz formula the number of ramification points of f is then

$$2(2 - 2g(\mathbb{P}^1)) - 2 - 2g(E) = 4.$$

Note that f can be replaced by $f' = \alpha f + \beta$ for any $\alpha, \beta \in k, \alpha \neq 0$. This corresponds to moving the ramification points by automorphism of $\mathbb{P}^1$ preserving ∞.

Conversely, any $2 : 1$ covering $\varphi : E \to \mathbb{P}^1$ can be obtained by composing f with some automorphism $\mathbb{P}^1 \to \mathbb{P}^1$. Indeed by Hurwitz formula φ has 4 ramification points. After composing with an automorphism of $\mathbb{P}^1$ we may assume one of them is ∞. The abusing the same notation φ for $\varphi : E \to \mathbb{A}^1$ we have

$$\mathrm{div}(\varphi) + 2[A] \geq 0$$

so $\varphi \in L(2[A])$ and therefore $\varphi = \alpha f + \beta$ for some $\alpha, \beta \in k, \alpha \neq 0$. This shows that φ and f are related by an automorphism of $\mathbb{P}^1$.

Theorem 2.1. *(1) Every elliptic curves E is isomorphic to a smooth cubic curve in $\mathbb{P}^2$.*

(2) If C is a smooth cubic curve in $\mathbb{P}^2$, it is a $\mathbb{P}\mathrm{GL}_3$-translate of a Weierstrass cubic

$$y^2 z - f(x, z) = 0$$

for some binary cubic form $f(x, z)$ with three distinct roots all different from $(1 : 0)$.

(3) A smooth cubic curve in $\mathbb{P}^2$ has genus 1.

(4) Two smooth cubics in $\mathbb{P}^2$ are isomorphism as abstract C_1 and C_2 are isomorphic as abstract curves if and only if they are $\mathbb{P}\mathrm{GL}_3$ translates.

PROOF. (1) Choose bases 1 in $L([A])$, $\{1, f\}$ in $L(2[A])$ and $\{1, f, g\}$ in $L(3[A])$. Then the seven functions

$$1, f, g, f^2, fg, f^3, g^2$$

in $L(6[A])$ are linearly dependent:

$$a_1 + a_2 f + \cdots + a_7 g^2 = 0.$$

In other words the map $\psi : E \to \mathbb{P}^2$ via $p \mapsto [1 : f(p) : g(p)]$ has its image in cubic curve C cut out by homogenization of the above equation.

Claim 1: $\deg(\psi) = 1$.

$$\begin{tikzcd}
 & k(E) \arrow[d, "\deg d"] \arrow[ddl, "\deg 2"'] \arrow[ddr, "\deg 3"] & \\
 & k(f,g) \arrow[dl] \arrow[dr] & \\
k(f) & & k(g)
\end{tikzcd}$$

Since d divides 2 and 3 we conclude that $d = 1$.

Claim 2: C is smooth. Assume $p \in C$ is a singular point. Project from p to any $\mathbb{P}^1 \subset \mathbb{P}^2$ which does not pass through p. This gives a degree one map $C \to \mathbb{P}^1$. Composing with $\psi : E \to C$ also of degree one we obtain a degree one map (i.e. a birational isomorphism $E \to \mathbb{P}^1$). This is impossible because $g(E) = 1$ and $g(\mathbb{P}^1) = 0$.

We conclude from the above two claims that ψ is an isomorphism between E and a smooth cubic curve C.

(2) We may assume that C passes through $A = (0 : 1 : 0)$ with is an inflection point of C, i.e. C has a 'double tangent' at A. In other words A is the intersection of C with the Hessian curve given by

$$\det\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right).$$

We can make a change of coordinates such that the tangent line at A is $z = 0$. Then the equation $F(x, y, z)$ of C has

(a) no y^3 term (because $A \in C$),

(b) no $y^2 x$ term (sing the tangent line at A is $z = 0$),

(c) no yx^2 term (because of the double tangency),

(d) a non-zero $y^2 z$ term (since $z = 0$ is the tangent line).

So

$$F = y^2z + yzf_1(x,z) + f_3(x,z)$$

where f_i is a binary form of degree i. After exchanging y by $y + \frac{1}{2}f_1(x,z)$ we can remove the middle term and get the equation

$$y^2z = f(x,z)$$

as desired. Since C is smooth, $f(x,z)$ cannot be divisible by z and $f(x,z)$ has distinct roots. If $(a:b)$ is a multiple root then $(a:0:b)$ is a singular point of C.

(3) In view of part (2) project from $A = (0:1:0)$ to the line $y = 0$. Tangent line to C at $p \in C$ is

$$\frac{\partial F}{\partial x}(p)x + \frac{\partial F}{\partial y}(p)y + \frac{\partial F}{\partial z}(p)z = 0$$

This passes through A if and only if $\frac{\partial F}{\partial y}(p) = 0$. Here $F = y^2z - f(x,y)$. Now observe that $\pi : C \to \mathbb{P}^1$ is a 2:1 covering with precisely four ramification points and the claim follows from Hurwitz formula.

(4) By part (2) assume that C_1 and C_2 are both in Weierstrass form

$$C_1 : y^2z = f_1(x,z)$$
$$C_2 : y^2z = f_2(x,z).$$

Suppose $\varphi_1 : E \to C_1$ and $\varphi_2 : E \to C_2$ are isomorphisms. Then, considering projections to the $(x:z)$ line we may assume $\varphi_1(A) = \varphi_2(A) = (0:1:0)$. Then there is $g \in \mathbb{P}\mathrm{GL}_2$ taking roots of f_1 to roots of f_2. This shows that C_1, C_2 are $\mathbb{P}\mathrm{GL}_3$-translates. Note that

$$\varphi_i^*(x/z) \in L(2[A]), \varphi_i(y/z) \in L(3[A]), \quad i : 1,2.$$

Writing

$$\varphi_2^*(x/z) = \alpha\varphi_1^*(x/z) + \beta, \text{ and } \varphi_2^*(y/z) = c_1\varphi_1^*(y/z) + c_2\varphi_1^*(x/z) + c_3.$$

We obtain an element of $\mathbb{P}\mathrm{GL}_3$ taking C_1 to C_2.

□

3. The moduli problem and its solution

Our original moduli problem is this:

$$\mathcal{F}_{1,1}(S) = \{(p : X \to S, A : S \to X) : \text{ satisfying (1), (2) and (3) below } \}$$

(1) $p : E \to S$ is a proper smooth morphism,

(2) $A : S \to E$ is a section,

(3) all geometric fibers of S are elliptic curves.

We introduce another functor

$$\mathcal{F}_1(S) = \{p : X \to S : \text{ satisfying (1), (2) and (3) below } \}$$

(1) $p : E \to S$ is a proper smooth morphism,

(2) There is an etale cover $\{U_i\}$ of S and a section $s_i : U_i \to X|_{U_i}$ over each U_i,

(3) all geometric fibers of S are elliptic curves.

The two functors coincide over a point but they are distinct in general. Now view $\mathbb{P}^9$ as the space of plane cubic curves in $\mathbb{P}^2$ and let $U \subset \mathbb{P}^9$ be the open subset of smooth cubic curves.

Theorem 3.1. *The family of all smooth cubic curves over $U \subset \mathbb{P}^9$ has an etale local universal property for $\mathcal{F}_1$ (i.e. the local universal property of 13 with the opens taken to be etale opens).*

Proof. The only thing that has remained to be proved for this assertion is to show that there is an etale cover of U with local sections. But this is given by intersecting with the Hessian curve. □

We conclude that

$$\mathcal{M}_1 = U /\!\!/ \mathrm{SL}_3$$

is a coarse moduli space for $\mathcal{F}_1$ if it is a categorical quotient and an orbit space.

Note that $U = (\mathbb{P}^9)^{ps}$ for aciton of SL_3 (or equivalently for $\mathbb{P}\mathrm{GL}_3$). Because of this $U /\!\!/ \mathrm{SL}_3$ is an orbit space. We know that

$$\dim(U /\!\!/ \mathrm{SL}_3) = \dim U - \dim \mathrm{SL}_3 = 1$$

and in fact

$$\mathcal{M}_1 = \mathbb{A}^1 = U /\!\!/ \mathrm{SL}_3 \subset (\mathbb{P}^9)^{ss} / \mathrm{SL}_3 = \mathbb{P}^1.$$

There is a moduli space for the moduli functor $\mathcal{F}_{1,1}$ as well. Indeed,

Theorem 3.2. *THe following family has a local universal property ofr $\mathcal{F}_{1,1}$ in Zariski topology.*

$$S = \{(a, b, c) : (4a^3 + 27b^2)c = 1\}.$$

Proof. There is a fibration

$$E = \{(x : y : z) : y^2 z = x^3 + axz^2 + bz^3\} \subset \mathbb{P}^2 \times S \xrightarrow{p} S$$

via $(x : y : z) \mapsto (x, y, z)$ and the section is $(a, b, c) \mapsto ((0 : 1 : 0), (a, b, c))$. Moreover $s = (a_1, b_1, c_1)$ and $t = (a_2, b_2, c_2)$ have isomorphic fibers

$$E_s : y^2 z = x^3 + a_1 x z^2 + b_1 z^3$$
$$\text{and } E_t : y^2 z = x^3 + a_2 x z^2 + b_2 z^3$$

if and only if

$$a_2 = t^2 a_1, b_2 = t^3 b_1$$

for some $t \in \mathbb{G}_m$. Hint: in fact if $\alpha : \mathbb{P}^2 \to P^2$ is the isomorphism of $(C_1, (0:1:0))$ and $(C_2, (0:1:0))$ given by

$$x \mapsto \alpha_{11}x + \alpha_{12}y + \alpha_{13}z$$
$$y \mapsto \alpha_{21}x + \alpha_{22}y + \alpha_{23}z$$
$$z \mapsto \alpha_{31}x + \alpha_{32}y + \alpha_{33}z$$

then $\alpha_{ij} = 0$ if $i \neq j$. So α has to take the tangent line $z = 0$ to C_1 at $(0:1:0)$ to the tangent line $z = 0$ to C_2 at $(0:0:1)$. Thus $\alpha_{31} = \alpha_{32} = 0$. Let A_1, A_2, A_3 be the point of order 2, the for $y/z \in L(3[A])$ we have

$$y/z(A_i) = 0, i = 1, 2, 3.$$

Thus

$$y/z \in L(3[A] - [A_1] - [A_3])$$

which is one-dimensional. Thus α preserves y/z up to constant.

Note that each fiber

$$y^2 z - (x^3 + axz^2 + bz^3) = 0$$

is a smooth cubic curve in $\mathbb{P}^2$. We can project this curve to $\mathbb{P}^1$ from $(0:1:0)$. The projection is given by

$$(x:y:z) \mapsto (x:z).$$

This is generically 2:1 with four ramification points over $(1:0)$ and the 3 roots of $x^3 + axz^2 + bz^3$. □

This time, our general machinery supplies us with a coarse moduli space

$$\mathcal{M}_{1,1} = S /\!\!/ \mathbb{G}_m.$$

In fact all points of S are property stable for our $\mathbb{G}_m$-action in $\mathbb{C}^3$. This is because (a,b,c) is unstable if and only if either

$$a = b = 0 \text{ or } c = 0$$

and properly stable otherwise. Thus $S /\!\!/ \mathbb{G}_m$ will be an orbit space (each fiber of quotient map $S \to S /\!\!/ \mathbb{G}_m$ is exactly one orbit of $\mathbb{G}_m$).

To realize $S /\!\!/ \mathbb{G}_m$, we work out the ring of invariants

$$k[S]^{\mathbb{G}_m} = k[a,b,c]^{\mathbb{G}_m} / ((4a^3 + 27b^2)c - 1).$$

The ring of invariants is generated as a module by monomials

$$\{a^\alpha b^\beta c^\gamma : 2\alpha + 3\beta - 6\gamma = 0\}.$$

It is easy to see then that this is the polynomial ring $k[a^3c, b^2c]$. So we have

$$k[S]^{\mathbb{G}_m} = k[a^3 c].$$

Remark. This invariant

$$a^3c = \frac{a^3}{4a^3 + 27b^2}$$

is called the j-invariant of $E : y^2z = x^3 + axz^2 + bz^3$. And we have a quotient map

$$S \xrightarrow{j} \mathbb{A}^1, \text{ via } (a, b, c) \mapsto a^3c.$$

CHAPTER 8

Construction of M_g

We give an outline of the construction of M_g when $g \geq 2$ in this chapter.

1. Hilbert polynomial

For $X \subset \mathbb{P}^n$, cut out by homogeneous ideal $I \subset R = k[x_0, \cdots, x_n]$, let

$$h_X(m) = \dim(R/I)_m.$$

THEOREM 1.1 (Hilbert). *There is a polynomial $p_X(m)$ such that*

$$h_X(m) = p_X(m)$$

for all large enough $m >> 0$.

REMARK. $p_X(m)$ carries a lot of information about $X \hookrightarrow \mathbb{P}^n$. In particular the leading term of $P_X(m)$ is $\frac{d}{s!}m^s$ where s is the dimension of X and d is the degree $\deg(X)$.

2. The Hilbert scheme

Consider the moduli problem that assigns to each scheme S, the set $\mathcal{F}(S)$ of flat proper families

$$X \subset \mathbb{P}^n \times S$$

over S such that $\pi^{-1}(s)$ has Hilbert polynomial $p(m)$ as a subvariety of $\mathbb{P}^n$.

THEOREM 2.1 (Grothendieck). *There exists a fine moduli space, $Hilb_{n,p}$ for $\mathcal{F}$.*

3. Our moduli problem

Now fix $g \geq 2$. Let C be a curve of genus g (smooth, projective). Choose a canonical divisor K on C. Let $D = 3K$ be the divisor. Then by Riemann-Roch

$$\deg(K) = 2g - 2 > 0, \deg(3K) = 6g - 6 \Rightarrow \ell(3K) = 5g - 5.$$

Let $f_0, \cdots, f_n$ be a basis of $L(3K)$, where $n = 5g - 6$. Then there is a corresponding Veronese embedding

$$\begin{aligned} \varphi : C &\to \mathbb{P}^n \\ p &\mapsto (f_0(p) : \cdots : f_n(p). \end{aligned}$$

The Hilbert polynomial of $\varphi(C)$ is

$$p(m) = (6m - 1)(g - 1).$$

So we can think of $\varphi(C)$ as a point of the Hilbert scheme $Hilb_{n,p}$, denoted by $[C, f_0, \cdots, f_n]$. Note that $\mathbb{P}\mathrm{GL}_{n+1}$ naturally acts on $\mathbb{P}^n$ and on $Hilb_{n,p}$, sending $[C, f_0, \cdots, f_n]$ to $[C, f_0', \cdots, f_n']$ for another basis of $L(3K)$.

THEOREM 3.1 (Mumford). *There existsa linear representation V of $\mathbb{P}\mathrm{GL}_{n+1}$ and a linearization*

$$Hilb_{n,p} \hookrightarrow \mathbb{P}(V)$$

of the $\mathbb{P}\mathrm{GL}_{n+1}$-action on $Hilb_{n,p}$ such that the image of $[C, f_0, \cdots, f_n]$ is properly stable in $\mathbb{P}(V)$ for any smooth C and any basis $f_0, \cdots, f_n$ of $L(3K)$ as above.

The GIT-quotient

$$H_g /\!\!/ \mathbb{P}\mathrm{GL}_{n+1} = M_g$$

is the coarse moduli space for the functor of curves of genus g. Here

$$H_g = \{[C, f_0, \cdots, f_n]\} \subset Hilb_{n,p}.$$

APPENDIX A

Toric varieties as GIT quotients

1. Review of toric varieties

We consider the case of action of $G = \mathbb{G}_m$ on $\mathbb{A}^n$ (which can be generalized to action of arbitrary tori on the affine space). Say the action is given via

$$t(x_1, \cdots, x_n) = (t^{a_1} x_1, \cdots, t^{a_n} x_n).$$

The induced action on coordinate ring is given on monomials $x^m = x_1^{m_1} \cdots x_n^{m_n}$ by

$$t.x^m = t^{\sum a_i m_i} x^m.$$

So the ring of invariant regular functions is generated by

$$\{x^m : \sum a_i m_i = 0\}.$$

So if we correspond the point $(m_1, \cdots, m_n) \in \mathbb{Z}^n_{\geq 0}$ to x^m, then $k[\mathbb{A}^n]^G$ corresponds to the lattice points

$$a^\perp \subset \mathbb{Z}^n_{\geq 0}.$$

Example 1.1. Let G act on $\mathbb{A}^3$ by the weight vector $a = (1, 1, -2)$. And the ring of invariants is generated by

$$m_1 = (2, 0, 1), m_2 = (0, 2, 1), m_3 = (1, 1, 1).$$

Say $u = x^{m_1}, v = x^{m_2}, w = x^{m_3}$. This way it is easy to see that

$$k[\mathbb{A}^n]^G = k[u, v, w]/(uv - w^2).$$

⌟

The invariant monomials of the coordinate ring form a subset of $M = \mathbb{Z}^m$, which give a real cone $C \subset \mathbb{R}^m$ that is a semigroup (actually a monoid). We let $k[C]$ denote the semigroup algebra

$$k[C] = \{ \sum_{\text{finite}} c_i x^m : c_i \in k, m_i \in C\}$$

Definition 17. $\mathcal{S}pec\, k[C]$ is an affine toric variety.

If $k[C]$ is graded by $\mathbb{Z}_{\geq 0}$, $\mathcal{P}roj\, k[C]$ is covered by affine toric varieties and is an example of *projective* toric varieties.

Example 1.2. Let C be the first quadrant cone on $\mathbb{R}^2$. Here $M = \mathbb{Z}^2$. On the integral lattice $C \cap M$, consider the grading

$$\deg : \mathbb{Z}^2 \to \mathbb{Z}; \quad (i, j) \mapsto i + 2j.$$

This induces a grading on the polynomial ring by

$$\deg x^m = \deg(m)$$

for any monomial x^m. Here is how to get the projective variety

$$\mathcal{P}roj\, k[C]$$

from this. It is obviously covered by $U_1 = \mathcal{S}pec(k[C][x^{-1}])_0$ and $U_2 = \mathcal{S}pec(k[C][y^{-1}])_0$. But combinatorially, let $e_1 = (1, 0)$ vector correspond to x and $e_2 = (0, 1)$ correspond to y. Let $D = \langle C \cup \{-e_1\}\rangle$ and let E be the line segment corresponding to the degree 0 piece in this upper half plane. Then

$$(k[C][x^{-1}])_0 = k[E]$$

and we can do the same thing with the half plane of first and fourth quadrants. The gluing is along

$$(k[C][x^{-1}, y^{-1}])_0 = k[u^{\pm 1}].$$

⌟

2. GIT quotients

Let G act on $\mathbb{A}^n$ and L be a line bundle. Then for any linearization of the action of G on L we have

$$\mathbb{A}^n /\!\!/ G = \mathcal{P}roj\, R, \quad R := \oplus_{d=0}^{\infty} \Gamma(X, L^{\otimes d})^G.$$

For instance let L be the trivial line bundle on $\mathbb{A}^n$, then the induced action of a one-parameter subgroup

$$t(x_1, \cdots, x_n) = (t^{a_1} x_1, \cdots, t^{a_n} x_n)$$

is by a choice of character t^b, and is explicitly given by

$$t.(x_1, \cdots, x_n, x_{n+1}) = (t^{a_1} x_1, \cdots, t^{a_n} x_n, t^b x_{n+1})$$

and the induced action on $\Gamma(X, L)$ is

$$(t.f)(p) = t^b f(t^{-1} p).$$

Lemma 17. *On monomial sections we have*

$$t.x^m = t^{b - \sum a_i m_i}.x^m \text{ for } x^m \in \Gamma(X, L)$$
$$t.x^m = t^{db - \sum a_i m_i}.x^m \text{ for } x^m \in \Gamma(X, L^{\otimes d}).$$

Corollary 8. $x^m \in \Gamma(X, L^{\otimes d})^G$ *if and only if* $db = \sum a_i m_i$.

Example 2.1. Say $\mathbb{G}_m$ is acting on $\mathbb{A}^3$ by $a = (1, 1, -2)$ and $b = 1$ from the above notation. Then the plane of

$$a.m = k$$

gives the monomials in the k-grade piece R_k. The grading

$$\deg : \mathbb{Z}^3 \to \mathbb{Z}, \quad m \mapsto m.a$$

gives the GIT-quotient $\mathcal{P}roj\, k[C] = \mathbb{A}^3 /\!\!/ G$.

⌟

APPENDIX B

Moduli of representations of a quiver

To do.

APPENDIX C

Comments on homework problems

Problem 1 of set 1. Probably the easiest proof is to use the incidence $X = \{(Q, L) : L \subset Q\}$ having projections to the space $\mathbb{P}^{\frac{n(n+1)}{2}-1}$ of all quadrics and to $\mathrm{Gr}(n, d)$. $\square$

Problem 1 of set 3. H acts on G. If $g_1, g_2 \in G$ then $Hg_1 \cong Hg_2$ iva $h \mapsto h(g_1^{-1}g_2)$. $f(H) = H.e$ in G. So it is enough to show that there is a closed H-orbit in G. Use noetherian induction for this: start with $x_! \in G$. If Hx_1 is not closed, then choose $x_2 \in \overline{Hx_1} \smallsetminus Hx_1$. If H_2x_2 is not closed, choose $x_3 \in \overline{Hx_2} \smallsetminus Hx_2$ and iterate. The sequence

$$\cdots \subsetneq \overline{Hx_3} \subsetneq \overline{Hx_2} \subsetneq \overline{Hx_1}$$

has to terminate. $\square$

Problem 4 of set 3. For the second part we have

$$\begin{aligned}\mathrm{Stab}_{\mathrm{SL}_2}(f_0) &= \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} : ad - bc = 1, (ax+by)^n = x^n\right\}\\ &= \left\{\begin{pmatrix} \zeta & 0 \\ c & \zeta^{-1}\end{pmatrix} : \zeta^n = 1\right\} = \mathbb{G}_a \rtimes \mu_n.\end{aligned}$$

We conclude that $\mathrm{Stab}_{\mathrm{SL}_2}(f_0)$ contains no $\mathbb{G}_m$. $\square$

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