Geometry of Hilbert schemes

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ABSTRACT. The goal of this course is to define/construct the Hilbert scheme. The applications are to Donaldson-Thomas invariants.

Contents

1. Representable functors

Here we develop an alternative point of view to schemes; that is we view schemes of functors (functor of points). This leads to construction of moduli spaces.

Convention: All rings are commutative with unit, and morphisms of rings map unit to unit. An R-algebra is a ring A with a ring map $R \to A$. Consequently, note that a Z-algebra is just any ring (for any ring R there is a unique morphism of rings $\mathbb{Z} \to R$).

Let X be a functor $X : R - alg \to Set$ via $A \mapsto X(A)$ and on morphism $A \xrightarrow{\varphi} B \mapsto X(A) \xrightarrow{\varphi_*} X(B)$ we call these functors presheaves.

EXAMPLE 1. View the affine scheme cut out by $y^3 = x^3 - x$ over $\mathbb{Z}$ by a functor

$$\begin{aligned} X : \mathbb{Z} - alg &\to Set \\ A &\mapsto X(A) = \{(x, y) \in A \times A : y^3 = x^3 - x\}. \end{aligned}$$

One can check that this is actually a functor. More generally let R be a ring and $f_1, \cdots, f_r \in R[x_1, \cdots, x_n]$. Then let X be the presheaf

$$X(A) = \{(x_1, \cdots, x_n) \in A^n : f_1(x_1, \cdots, x_n) = f_2(x_1, \cdots, x_n) = \cdots = f_r(x_1, \cdots, x_n) = 0\}$$

this is a presheaf and in fact this functor is the affine scheme defined by $f_1, \cdots, f_n$. This functor is in fact representable by $Z(f_1, \cdots, f_r) \subset \mathbb{A}^n_R$.

One interesting special case is when $r = 0$. Then $X = \mathbb{A}^n_R$:

$$\mathbb{A}^n_R(A) = \{(x_1, \cdots, x_n) \in A^n\}$$

and if $r \neq 0$ then X is a subfunctor of $\mathbb{A}^n_R$.

More exclusively for an R-algebra A we correspond the functor $\mathcal{S}pec\, A$ given by

$$\begin{aligned} \mathcal{S}pec\, A : R - alg &\to Set \\ B &\mapsto \mathcal{S}pec\, A(B) = \mathrm{Hom}_R(A, B) \end{aligned}$$

This is a functor representable by A (i.e. it is $h_{\mathcal{S}pec\, A}$). Recall that an isomorphism $X \cong h_{\mathcal{S}pec\, A}$ is given by a *universal object* $x_0 \in X(A)$.

DEFINITION 1. An affine scheme over R is a representable presheaf $X : R - alg \to Set$ i.e. there exists an R-algebra A and $x_0 \in X(A)$ representing X.

REMARK. The category of presheaves with morphisms being natural transformations among them, admits fibered products. Here is a construction: given

$$\begin{array}{ccc} & & Z \\ & & \downarrow f \\ X & \xrightarrow[f]{} & Y \end{array}$$

define $W : R - alg \to Set$ via $A \mapsto X(A) \times_{Y(A)} Z(A)$ is the fibered product promised. This complete the diagram to a square

$$\begin{array}{ccc} W & \longrightarrow & Z \\ \downarrow & & \downarrow f \\ X & \xrightarrow[f]{} & Y \end{array}$$

which satisfies the universal mapping property in the category of presheaves.

Remark. Our most interesting case if when $R = \mathbb{C}$ and then $X(\mathbb{C})$ is the set of $\mathbb{C}$-valued points of X.

2. Projective space

We know that

$$\mathbb{P}^n : R - alg \to Set$$

is given by

$$\mathbb{P}^n(A) = \{(L, s_0, \cdots, s_n) : L \text{ is an invertible } A\text{-module}, s_0, \cdots, s_n \in L \text{ such that } L = \sum_{i=0}^{n} As_i\}/\sim .$$

and on mappings we have

$$A \to B \mapsto (L, s_0, \cdots, s_n) \mapsto (L \otimes_A B, s_0 \otimes_A 1, \cdots, s_n \otimes_A 1).$$

Definition 2. An A-module L is invertible if for any prime ideal $\mathfrak{p}$ of A, the localization $L_\mathfrak{p}$ over $A_\mathfrak{p}$ is free of rank one (i.e. there is $s \in L_\mathfrak{p}$ such that $A_\mathfrak{p} \to L_\mathfrak{p}$ via $a \mapsto as$ is an isomorphism).

Note 2.1. If L is invertible over A and $\varphi : A \to B$ is a morphism, and is a prime ideal of B, then

$$(L \otimes_A B)_\mathfrak{q} = L \otimes_A B \otimes_B B_\mathfrak{q} = L \otimes_A B_\mathfrak{q} = L \otimes_A A_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{q} = L_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{q}$$

and $L_\mathfrak{p} \otimes_{A_\mathfrak{p}} B_\mathfrak{q}$ is free of rank 1 on $B_\mathfrak{q}$. Thus $L \otimes_A B$ is invertible over B if L is invertible over A. ⌟

Definition 3. The isomorphism of the above data is given by $(L, s_0, \cdots, s_n) \cong (M, t_0, \cdots, t_n)$ if the is an isomorphism of A-modules $\varphi : L \to M$ such that $\varphi(s_i) = t_i$ for all i.

Example 2. If $A = k$ is a field, an invertible module over it is just a 1-dimensional vector space. Thus

$$\mathbb{P}^n(k) = \{(s_0, \cdots, s_n) \in k^{n-1} : s_0, \cdots, s_n \in k \text{ such that not all are zero }\}/k^*$$

by a canonical isomorphism.

3. Closed immersions/subpresheaves

Let's first consider the affine case: Let A be an R-algebra and $I \subset A$ is an ideal of it. The projection $A \to A/I$ induces a morphism of affine schemes $\mathit{Spec}\, A/I \to \mathit{Spec}\, A$ and consequently a morphism $\mathrm{Hom}_R(A/I, .) \to \mathrm{Hom}_R(A, .)$ by composition. We can regard $\mathrm{Hom}_R(A/I, .)$ as a subfunctor of $\mathrm{Hom}_R(A, .)$: any morphism $A/I \to B$ is a morphism $A \to B$ that vanishes on I, hence for all B, $\mathit{Spec}\, A/I(B)$ is a subset of $\mathit{Spec}\, A(B)$ so it is a subpresheaf of $\mathit{Spec}\, A$.

Now let $f : Y \to X$ be a morphism of presheaves/R.

Definition 4. f is a closed immersion if for every R-algebra A and every $x \in X(A)$ there is an ideal $I \subset A$ such that

$$\begin{array}{ccc} \mathit{Spec}\, A/I & \longrightarrow & \mathit{Spec}\, A \\ \big\downarrow & & \big\downarrow x \\ Y & \xrightarrow[f]{\qquad} & X \end{array}$$

is a fibered product.

Example 3. Let $f = zy^2 - x^3 + z^2x \in R[x, y, z]$ be the homogeneous polynomial given. Define $Z(f) \subset \mathbb{P}^2$ by $(L, x, y, z) \in \mathbb{P}^2(A)$ such that $zy^2 - x^3 + z^2x \in L^{\otimes 3}$ in zero. Check that $Z(f) \subset \mathbb{P}^2$ is a closed immersion.

Note 3.1. Recall that $\mathrm{GL}_m : R - alg \to Set$ via $A \mapsto \mathrm{Gl}_n(A)$ is a representable functor and the special case of $m = 1$ is denoted by $\mathbb{G}_m$. Besides, the additive group object $\mathbb{G}_a$ is the functor $A \mapsto A^+$. Recall also the notion of action of a group presheaf G on presheaf X. As an example we have the action of $\mathbb{G}_m$ on $\mathbb{A}^{n+1} - \{0\}$ where

$$\mathbb{A}^{n+1} - \{0\} : A \mapsto \{(a_0, \cdots, a_n) \in \mathbb{A}^{n+1} : \langle a_0, \cdots, a_n \rangle = 1\}$$

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More generally let $I \subseteq A$ be an ideal, the corresponding closed subscheme is $\mathit{Spec}\, A/I \to \mathit{Spec}\, A$ when

$$\mathit{Spec}\, A/I(B) = \{A \to B : IB = 0\}$$

and its *open complement* is given by

$$U(B) = \{A \to B : IB = B\}.$$

Example 4. $\mathbb{G}_m(A)$ acts of $\mathbb{A}^{n+1} - \{0\}(A)$ by

$$\lambda.(a_0, \cdots, a_n) \mapsto (\lambda a_0, \cdots, \lambda a_n)$$

also we have a morphism of presheaves $\mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ given by

$$(a_0, \cdots, a_n) \mapsto (A, a_0, \cdots, a_n)$$

Note that the automorphism group of A and A-module is the same as A^* i.e. the group of units of A. Thus for all A, we have an injection

$$\mathbb{A}^{n+1} - \{0\}(A)/\mathbb{G}_m(A) \hookrightarrow \mathbb{P}^n(A)$$

which is not always surjective.

In fact, let Y be any affine variety over an algebraically closed field k. Let $A = \mathcal{O}(Y)$ be the affine coordinate ring which is a finitely generate k-algebra but an infinite dimensional k-module. Let K be the quotient field of Y. Then all localizations are contained canonically in K, $A \hookrightarrow K$. If K is an invertible A-module then $L \to L \otimes_A K$ is an injection, and $L \otimes_A K$ is a 1-dimensional K vector space hence isomorphic to L. So think of L as an A-submodule of K.

If $L = \langle m_1, \cdots, m_n \rangle$, $m_i \in K$, multiplying by common denominator we get L as a sub-module of A. So every finitely generated invertible A-module is isomorphic to an ideal of A, which is locally principal. We conclude that

$$\mathbb{P}^n(A) = \{(L, a_0, \cdots, a_n) : L \subset K \text{ as an } A\text{-submodule, locally principal }\}/A^*.$$

▸ EXERCISE 1. (1) $Y = \mathcal{S}pec(\mathbb{C}[x,y]/(y^2 - x^3 + x))$ is an affine elliptic curve. Then $(x,y) \subset A$ is a locally principle ideal but not principle.

(2) Let $A = \mathbb{C}[x,y]$ and $Y = \mathbb{A}^2$ then (x,y) is not locally principle.

By uniqueness of extension $\mathrm{Hom}(A_f, B) \to \mathrm{Hom}(A, B)$ is injective. The image is all $\varphi : A \to B$ such that $\varphi(f) \in B^*$. Think of $\mathcal{S}pec\, A_f \hookrightarrow \mathcal{S}pec\, A$ as a subfunctor: Say $F : Sch/k \to Set$ via $X \mapsto \mathrm{Hom}(X, \mathcal{S}pec\, A)$ then $G : Sch/k \to Set$ via $X \mapsto \mathrm{Hom}(X, \mathcal{S}pec\, A_f)$ is a subfunctor of it.

$$\begin{array}{ccccc}
\mathcal{S}pec\, A/(f) & \xhookrightarrow{q^\#} & \mathcal{S}pec\, A & \hookleftarrow & \mathcal{S}pec\, A_f \\
 & & \uparrow{\scriptstyle \varphi^\#} & & \\
 & & \mathcal{S}pec\, k & &
\end{array}$$

for any field k and any $\varphi : A \to k$. Then $\varphi(f) = 0$ if $\varphi^\#$ factors through $\mathcal{S}pec\, A/(f)$ and $\varphi(f)$ is invertible if $\varphi^\#$ factors through $\mathcal{S}pec\, A_f$. This justifies to a certain extent calling $\mathcal{S}pec\, A_f$ the open complement of $\mathcal{S}pec\, A/(f)$. In fact $\mathcal{S}pec\, A_f \hookrightarrow \mathcal{S}pec\, A$ is an open immersion and it is affine.

Not all open immersions are affine:

Example 5. $\mathbb{A}^n = \{0\} \to \mathbb{A}^n$ is not affine. Because the subfunctor

$$\begin{array}{ccc} \mathbb{A}^n - \{0\} \overset{}{\underset{q^\#}{\hookrightarrow}} & \mathbb{A}^n & \hookleftarrow * = \mathcal{S}pec\, R \\ & \uparrow \varphi^\# & \\ & \mathcal{S}pec\, k & \end{array}$$

Then for all $\varphi : R[x_1, \cdots, x_n] \to k$, if exists $\varphi(x_i) \neq 0$ then factors through $\mathbb{A}^n - \{0\}$ and if $\varphi(x_i) = 0$ for all i then factors through 0. In case $R = k$, algebraically closed field, if Y is an affine variety, then the morphism of varieties $Y \to \mathbb{A}^n$ factors through $\mathbb{A}^n - \{0\}$ if and only if $\forall \rho \in Y$, $a_i(\rho) \neq 0$ for any i. $\Rightarrow \langle a_1, \cdots, a_n \rangle$ is contained in no maximal ideal. $\Rightarrow \langle a_1, \cdots, a_n \rangle = A$ motivating the definition of $\mathbb{A}^n - \{0\}$.

?

Definition 5. $f : X \to \mathcal{S}pec\, A$ is an open immersion iff

(1) $X(B) \to \mathcal{S}pec\, A(B)$ is injective for all B (i.e. f is a mono)

(2) There exists an ideal $I \subset A$ such that $\varphi \in \operatorname{Hom}(A, B)$ is in $X(B)$ iff $\varphi(I)B = B$.

Example 6. $\mathbb{A}^n - \{0\} \hookrightarrow \mathbb{A}^n = \mathcal{S}pec[x_1, \cdots, x_n]$ is an open immersion and $I = (x_1, \cdots, x_n)$.

Example 7. $\mathcal{S}pec\, A_f \to \mathcal{S}pec\, A$ is an open immersion $I = (f)$.

Definition 6. $f : X \to Y$ is an open immersion iff

(1) $X(B) \to Y(B)$ is injective for all B (i.e. f is a mono)

(2) $\forall A, y \in Y(A)$ the fibered product $U \to \mathcal{S}pec\, A$ is an open immersion from

$$\begin{array}{ccc} U & \longrightarrow & \mathcal{S}pec\, A \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Remark. The definitions (1) and (2) agree if Y is affine: Given $\varphi : B \to A$, $U \hookrightarrow \mathcal{S}pec\, B$ is the complement of $\mathcal{S}pec\, B/f \hookrightarrow \mathcal{S}pec\, B$ then $V = U \times_{\mathcal{S}pec\, B} \mathcal{S}pec\, A$ is the complement of $\mathcal{S}pec\, A/IA \hookrightarrow \mathcal{S}pec\, A$.

Remark. If A is an R-algebra, $f_1, \cdots, f_n \in A$ such that $(f_1, \cdots, f_n) = A$ then $\mathcal{S}pec\, A_f \subset \mathcal{S}pec\, A$ form an open cover in the sense that every $\mathcal{S}pec\, k \overset{\varphi^\#}{\to} \mathcal{S}pec\, A$ for any field k factors through at least one $\mathcal{S}pec\, A_{f_i} \subset \mathcal{S}pec\, A$; i.e. $\varphi : A \to k$ can't map each f_i to zero, otherwise $\varphi(A) = 0$ but we know that $\varphi(1) = 1$.

Remark. (1) The previous definition is actually correct only for noetherian schemes. The correct definition is as follows: Let L be an A-module, is invertible if $\exists f_1, \cdots, f_n \in A$ such that $(f_1, \cdots, f_n) = 1$ (or elementary open covering $\{\mathcal{S}pec\, A_{f_i} \subset \mathcal{S}pec\, A\}_{i=1,\cdots,n}$) such that $\forall i = 1, \cdots, n$ L_{f_i} free rank 1 over A_{f_i}.

(2) If $i : Z \hookrightarrow X$ is a closed immersion, define the subfunctor $U \subset X$ by $U(A) = \{x \in X(A) : I_{(x)}A = A\}$ (this comes from definition of closed immersion). Then $U \hookrightarrow X$ is an open immersion and ideal $I \subset A$ for which "$IB = B$ iff $I^nB = B$" defines open the open complement. It is not clear that every open immersion has a closed complement.

LEMMA 1 (Baby descent theory). *$\langle f_1, \cdots, f_n\rangle = A$ and M be an A-module. Then*

$$M \overset{\gamma}{\hookrightarrow} \prod_i M_{f_i} \overset{\alpha,\beta}{\rightrightarrows} \prod_{i,j} M_{f_i f_j}$$

where $M_{ab} = (M_a)_b$, and $\alpha(m)_{i,j}$ is the image of $m_i \in M_{f_i f_j}$ and $\beta(m)_{i,j}$ is the image of m_j in $M_{f_i f_j}$. Then

(1) *γ is injective*

(2) *If $\alpha(x) = \beta(x)$ then $x \in \operatorname{im}(\gamma)$.*

that is the sequence is exact.

Note that we have the commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & M_a \\ \downarrow & & \downarrow \\ M_b & \longrightarrow & M_{ab} \end{array}$$

THEOREM 3.1. *$\mathbb{A}^n \to \mathbb{P}^n$ via*

$$\mathbb{A}^n(A) \to \mathbb{P}^n(A)$$
$$(a_1, \cdots, a_n) \mapsto (A, 1, a_1, \cdots, a_n)$$

is an open immersion and is the complement of the closed immersion $Z(x_0) \hookrightarrow \mathbb{P}^n$ given by

$$Z(x_0)(A) = \{(L, \ell_0, \cdots, \ell_n) \in \mathbb{P}^n(A) : \ell_0 = 0\}.$$

PROOF. **TO DO** ☐

4. $\mathcal{O}_X$-modules

DEFINITION 7. Let X be a presheaf. A presheaf M together with

(1) $M \overset{\pi}{\to} X$ is a morphism.

(2) $. : \mathbb{A}^1 \times M \to M$ such that

$$\begin{array}{ccc} \mathbb{A}^1 \times M & \overset{\cdot}{\longrightarrow} & M \\ & \underset{\pi\circ\text{proj}}{\searrow} & \downarrow \pi \\ & & X \end{array}$$

(3) $+ : M \times_X M \to M$ such that

$$\begin{array}{ccc} M \times_X M & \xrightarrow{+} & M \\ & \searrow & \downarrow \pi \\ & & X \end{array}$$

(4) $+ : M(A) \times_{X(A)} M(A) \to M(A)$ such that

$$\begin{array}{ccc} M(A) \times_{X(A)} M(A) & \xrightarrow{+} & M(A) \\ & \searrow & \downarrow \pi \\ & & X(A) \end{array}$$

get induced maps $A \times M_x \to M_x$ and $M_x \times M_x \to M_x$ making M_x an A-module. The induced map comes from $\mathrm{Hom}(\mathcal{S}pec\, A, \mathbb{A}^1) \cong \mathrm{Hom}(k[x], A) \overset{?}{\cong} A \times M_x \to M_x$

satisfying

(I) For any A and $x \in X(A)$ the set

$$M_x = \{m \in M(A) : m \mapsto x \in X(A) \text{ under } \pi\}$$

we have

$$\begin{array}{ccc} \mathbb{A}^1(A) \times M(A) & \xrightarrow{\times} & M(A) \\ & \searrow^{\pi \circ \mathrm{proj}} & \downarrow \pi \\ & & X(A) \end{array}$$

(II) For every $\varphi : B \to A$ and equivalently

$$\begin{array}{ccc} \mathcal{S}pec\, A & \xrightarrow{\varphi^\#} & \mathcal{S}pec\, B \\ & \searrow^{\varphi_* y} & \downarrow y \\ & & X \end{array}$$

get map $M_y \to M_{\varphi_* y}$ which is B-linear.

Such an object is called an $\mathcal{O}_X$-module.

Definition 8. In (II) get induced morphism of A-modules $M_y \otimes_B A \to M_{\varphi_* y}$. If this is an isomorphism for all φ and y, then M is called quasi-coherent.

Example 8. In the case $M = \mathbb{A}^1 \times X$, and $M \to X$ is the projection on second factor, then $\mathbb{A}^1 \times M \to M$ is the mapping

$$A \times A \times X \to A \times X$$
$$(a, b, x) \mapsto (ab, x)$$

and the product is the mapping

$$(\mathbb{A}^1 \times X) \times_X (\mathbb{A}^1 \times X) \to \mathbb{A}^1 \times X$$
$$((a,x),(b,x)) \mapsto (a+b,x)$$

Then $M_x = \mathbb{A}^1 \times \{x\}$, more generally $\mathcal{O}_X^{\oplus n}$ is given by

$$M = \mathbb{A}^n \times X \to X.$$

(*** $\mathbb{A}^1$ action on $\mathbb{A}^n$ etc.)

REMARK. For $\mathcal{O}_X^{\oplus n} = \mathbb{A}^n \times X$ the structure map is

$$\mathbb{A}^n \times X \overset{\text{proj}}{\to} X$$

$$\begin{array}{ccc} \mathbb{A}^n \times X & \xrightarrow{\text{proj}} & X \\ \uparrow & & \uparrow x \\ \mathcal{S}pec\, A[x_1,\cdots,x_n] & \xrightarrow{a} & \mathcal{S}pec\, A \end{array}$$

(*** this diagram appears without explanation!)

REMARK. The interesting case: affine X and coherent $\mathcal{O}_X$-module

Let A be an R-algebra and M is an A-module, then there is an A-algebra S and module map $\psi : M \to S$ such that for any A-algebra B

$$\mathrm{Hom}_{A-alg}(S,B) \to \mathrm{Hom}_{A-mod}(M,B)$$
$$\varphi \mapsto \varphi \circ \psi$$

is bijective. In fact we shall take S to be the symmetric product

$$\mathrm{Sym}_A M = \oplus_{n\geq 0} S_A^n M = \oplus_{n\geq 0} M^{\otimes n}/(x \otimes y - y \otimes x).$$

This is because for any $\alpha : A \to B$ a map of A-modules, there is a unique extension of α to $\mathrm{Sym}_A(M) \overset{\widetilde{\alpha}}{\to} B$ given by

$$\widetilde{\alpha}(m_1 \cdots m_n) = \alpha(m_1)\cdots\alpha(m_n).$$

EXAMPLE 9. $M = A^n$, $\mathrm{Hom}_{A-alg}(A[x_1,\cdots,x_n],B) = B^n = \mathrm{Hom}_{A-mod}(A^n,B)$ therefore

$$\mathrm{Sym}_A(A^n) \cong A[x_1,\cdots,x_n].$$

Given A, M as above we have a natural morphism $\mathcal{S}pec\,\mathrm{Sym}M \to \mathcal{S}pec\, A$. Let $\mathcal{M} = \mathcal{S}pec\,\mathrm{Sym}M$ and $X = \mathcal{S}pec\, A$. We now show that $\mathcal{M}$ is an $\mathcal{O}_X$-module. We need a morphism $\mathbb{A}^1 \times \mathcal{S}pec\,\mathrm{Sym}M \to \mathcal{S}pec\,\mathrm{Sym}M$, i.e. a morphism

$$\mathrm{Sym}M \overset{\pi}{\to} \mathrm{Sym}M \otimes k[x]$$

which is given naturally. Let $y : \mathcal{S}pec\, B \to \mathcal{S}pec\, A$ be an element of $X(B)$. Then

$$\mathcal{M}_y = \{m : \mathcal{S}pec\, B \to \mathcal{M} : \pi \circ m = y\}.$$

So if $y : \varphi^{\#} : \mathcal{S}pec\, B \to \mathcal{S}pec\, A$ then

$$\mathcal{M}_{\varphi^{\#}} = \mathrm{Hom}_{A-alg}(\mathrm{Sym}_A M, B).$$

And this turns $\mathcal{M} \to \mathcal{S}pec\, A$ into an $\mathcal{O}_{\mathcal{S}pec\, A}$-module. We also desire that $\mathcal{M} \to X$ is quasi-coherent. So we need for any mapping $A \to B$ that

$$\mathrm{Hom}_{A-mod}(M, A) \otimes_A B \to \mathrm{Hom}_{A-mod}(M, B) \tag{4.1}$$

be an isomorphism. BUT this is not true for an arbitrary A-module M.

Definition 9. AN A-module N is projective if for any epimorphism $\pi : N \to M$ of A-modules there exists a section $s : M \to N$.

Example 10. Free modules are projective!

Example 11. Invertible modules are projective.

Proof. **TO DO** □

If M is a projective finitely generated module then the mapping 4.1 is an isomorphism.

Step 1. First we prove this when M is free of finite rank.

$$A^{\oplus n} \otimes_A B \cong \mathrm{Hom}_A(A^{\oplus n}, A) \otimes_A B \to \mathrm{Hom}_A(A^{\oplus n}, B) \cong B^{\oplus n}.$$

Step 2. Say M is finitely generated and projective, then we have a surjection $A^{\oplus n} \to M \to 0$ and there is a section of it $s : M \to A^{\oplus n}$. But we can also complete the sequence with its kernel K by

$$0 \to K \to A^{\oplus n} \to M \to 0$$

and the fact that this short exact sequence is split, implies that $M \oplus K \cong A^{\oplus n}$. We know from previous step that $\mathrm{Hom}_A(A^{\oplus n}, A) \otimes_A B \cong \mathrm{Hom}_A(A^{\oplus}, B)$ therefore

$$\mathrm{Hom}_A(M \oplus K, A) \otimes_A B \cong \mathrm{Hom}_A(M \oplus K, B)$$

But both sides split

$$(\mathrm{Hom}_A(M, A) \otimes B) \oplus (\mathrm{Hom}_A(K, A) \otimes B) \cong \mathrm{Hom}_A(M, B) \oplus \mathrm{Hom}_A(K, B)$$

the result follows because the isomorphism is induces by direct sum of maps as in **??**.

Definition 10. The $\mathcal{O}_X$-module $M \to X$ is a vector bundle if $M \to X$ is affine and $\forall A, x \in X(A)$ there is a projective A-module P of finite rank making the following diagram cartesian:

$$\begin{array}{ccc} M & \longrightarrow & X \\ \uparrow & & \uparrow x \\ \mathcal{S}pec\, \mathrm{Sym} P & \longrightarrow & \mathcal{S}pec\, A \end{array}$$

If $M \to X$ is a vector bundle, then for any $A, x \in X(A)$,

$$\begin{aligned} M_x = &\operatorname{Hom}_{A-alg}(\operatorname{Sym} P, A) \qquad \text{, i.e. sections } \mathcal{Spec}\, A \to \mathcal{Spec}\, \operatorname{Sym} A \\ &\operatorname{Hom}_{A-mod}(P, A) =: P^\vee \end{aligned}$$

Example 12 (Eventual example). $\mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ is a vector bundle.

Proof. **TO DO** □

5. Hilbert schemes

Suppose X is a quasi-projective scheme, i.e. $X \overset{cl.im.}{\hookrightarrow} Z \overset{op.im.}{\hookrightarrow} \mathbb{P}^n$.

$$Hilb(X)(A) = \{ \text{ closed subschemes } Z \text{ of } X \times \mathcal{Spec}\, A, \text{ flat over } \mathcal{Spec}\, A\}$$

in particular if k if a field, then

$$Hilb(X)(k) = \{ \text{ closed subschemes } Z \subset X\}.$$

We can complete to a diagram

$$\begin{array}{ccccc} Z & \underset{\text{cl. imm.}}{\hookrightarrow} & X \times \mathcal{Spec}\, A & \longrightarrow & \mathcal{Spec}\, A \\ \downarrow & & \downarrow & & \downarrow Z \\ \mathfrak{Z} & \longrightarrow & X \times Hilb(X) & \longrightarrow & Hilb(X) \end{array}$$

using $\mathfrak{Z}$ such that both squares are cartesian.

$$\begin{aligned} \mathfrak{Z}(A) = \{(s, Z) : Z \subset X \times \mathcal{Spec}(A) &\text{ is a closed immersion and} \\ s : \mathcal{Spec}\, A \to Z \text{ a morphism such that the following is commutative.}\} \end{aligned}$$

$$\begin{array}{ccc} Z & \overset{s}{\longleftarrow} & \mathcal{Spec}\, A \\ \downarrow & \nearrow & \\ X \times \mathcal{Spec}\, A & & \end{array}$$

We can think of a close subscheme $Z \subset X \times \mathcal{Spec}\, A$ as a family of subschemes of X parametrized by $\mathcal{Spec}\, A$:

$$\begin{array}{ccccc} Z_t & \underset{\text{cl. imm}}{\longrightarrow} & X & \longrightarrow & \mathcal{Spec}\, k \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & X \times \mathcal{Spec}\, A & \longrightarrow & \mathcal{Spec}\, A \end{array}$$

6. Flatness

Flatness is the rigorous mathematical expression for the notion of a "continuously varying family".

DEFINITION 11. A is a ring and M is an A-module. M is said to be flat if the functor $M \otimes_A -$ is exact.

Note that tensor product is right-exact, hence we only need to check left-exactness. In particular, if $I \subset A$ is an ideal, we have an exact sequence

$$0 \to I \to A \to A/I \to 0$$

and if M is flat then

$$0 \to I \otimes_A M \to M \to M/IM \to 0$$

is exact. Hence $I \otimes_A M \to IM$ is an isomorphism. Conversely if this is true for all $I \subset A$ then M is flat.

EXAMPLE 13. Free modules are flat. Projective modules are also flat (since they are direct sums of free modules). Every localization $S^{-1}A$ is flat.

PROPOSITION 1. *M is flat over A iff M_m if flat over A_m for any maximal ideal $m \subset A$.*

PROOF. **TO DO** □

LEMMA 2. *Let M be a flat A-module. Then every linear relation $\sum_{i=1}^{n} a_i x_i = 0$ in M ($a_i \in A, x_i \in M$) can be lifted to some free module of finite rank. i.e. there exists A^s, φ, b_{ij} such that $A^s \xrightarrow{\varphi} M$ via $e_i \mapsto y_i$, $x_i = \sum_j b_{ij} y_j$ for all i and $\sum a_i b_{ij} = 0$.*

PROOF. Matsumura 2, Ch. 2, Section 3 [ey too rooooohet joghd!] □

COROLLARY 1. *If A is an integral domain, then M is torsion-free if it is flat.*

PROOF. If M has torsion element x, i.e. $ax = 0$ for some $a \neq 0$, then by the lifting property, there is $b_1, \cdots, b_r \in A$ and $y_1, \cdots, y_r \in M$ such that $\sum b_j y_j = x$ and $ab_j = 0$ for all j. But if $a \neq 0$ then $b_j = 0$ for all j, thus $x = 0$. □

From this we learn that:

COROLLARY 2. *If A is a local ring and M is a finitely generated A-module then M is flat iff M is free. In some sense flatness is a globalization of being free!*

For the prove we recall

LEMMA 3 (Nakayama's lemma). *Let I be an ideal in R, and M a finitely-generated module over R. If $IM = M$, then there exists an $r \in R$ with $r \cong 1(\mod I)$, such that $rM = 0$.*

Proof of Corollary. (Matsumura (2), Ch. 2, Section 3) We only need to show that flatness implies being free. In fact it suffices to show that for a minimal set of generators $x_1, \cdots, x_n$ of M, they form a basis as well. First we need to show that $x_1, \cdots, x_n \in M$ are such that their images $\overline{x_1}, \cdots, \overline{x_n} \in M/\mathfrak{m}$ are linearly independent over k then they are linearly independent over A. For this use the previous lemma together with induction on n. Secondly, we should show that if $\overline{x_1}, \cdots, \overline{x_n}$ generate $M/\mathfrak{m}$ then their lifts $x_1, \cdots, x_n$ generate M. If $r \in M$ is not generated by x_i's then rM is an A module such that $\mathfrak{m}(rM) = 0$ and thus by Nakayama's lemma $rM = 0$, contradicting it's nontriviality. □

Definition 12. An A-algebra is flat if it is flat as an A-module.

Lemma 4 (Support of M is closed!). *Let A be a ring, M a finitely generate A-module. Let $\mathfrak{p}$ be a prime ideal, such that $M_\mathfrak{p} = 0$. Then there exists $s \notin \mathfrak{p}$ such that $M_s = 0$.*

Note 6.1. We are not going to use noetherian condition! ⌟

Proof. Let $m_1, \cdots, m_n$ generate M. Then $\frac{m_1}{1}, \cdots, \frac{m_n}{1} \in M_\mathfrak{p}$ are all zero! Thus there are $s_1, \cdots, s_n \notin \mathfrak{p}$ such that $s_i m_i = 0$ in M. Let $s := \prod s_i$, then $sm_i = 0$ for all i. Also note that $s \notin \mathfrak{p}$. Hence $sm = 0$ for all $m \in M$ and $M_s = 0$. □

Proposition 2. *Let A be a ring and M and A-module. Let $n \in \mathbb{Z}$ be an integer. Then the following are equivalent:*

(1) *M is Zariski-locally free of rank n (i.e. there are $f_1, \cdots, f_r \in A$ such that $\langle f_1, \cdots, f_r \rangle = 1$ with M_{f_i} free of rank n over A_{f_i} for all i).*

(2) *M is finitely generated and at every maximal ideal $M_\mathfrak{m}$ is free of rank n.*

(3) *M is finitely generated, flat and for all maximal ideal $\mathfrak{m}$, $M/\mathfrak{m}M$ is n-dimensional vector space over $A/\mathfrak{m}$.*

Remark. The tensor product of noetherian rings is *not* necessarily noetherian. That is why we insist on not using the noetherian conditions. As an example not that $\mathbb{R} \otimes_\mathbb{Q} \mathbb{R}$ is not noetherian.

If we restrict to noetherian rings, then the fiber product of affine schemes may not be affine. This is inconvenient: it forces you to restrict to certains kinds of fiber products: at best we have under Hilbert's basis theorem that if A, B are noetherian and B is a finitely-generated A-algebra then $B \otimes_A C$ is noetherian.

As a result restriction to noetherians, forces restriction to affine morphisms of finite type (i.e. morphisms $X \to Y$ such that for all cartesian diagrams

$$\begin{array}{ccc} \mathcal{S}pec\, B & \longrightarrow & \mathcal{S}pec\, A \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

PROOF. **TO DO** $\square$

REMARK. The scheme $\mathcal{S}pec\, A_{\mathfrak{m}}$ can be thought of as the germ of $\mathcal{S}pec\, A$ at $\mathfrak{m}$ and $\mathcal{S}pec\, A/\mathfrak{m}$ as the point at $\mathfrak{m}$. The above is a passage between information at a point and information of an infinitesimal neighborhood. We can conclude that the vector bundles over $\mathcal{S}pec\, A$ are in one-to-one correspondence with the finitely generated flat modules such that all fibers are vector spaces of some dimension.

7. Hilbert scheme of n points

$Hilb^n(X)(A) = \{$ closed subscheme $Z \subset X \times \mathcal{S}pec\, A$ with the following properties holding for it $\}$

Properties: That $Z \to \mathcal{S}pec\, A$ is flat and finite. And for every maximal ideal $\mathfrak{m}$ of A with t corresponding to the $A/\mathfrak{m}$-valued point of $\mathcal{S}pec\, A$ and Z_t for the fiber product,

$$\begin{array}{ccc} Z_t & \longrightarrow & \mathcal{S}pec\, A/\mathfrak{m} \\ \downarrow & & \downarrow t \\ Z & \longrightarrow & \mathcal{S}pec\, A \end{array}$$

then $Z_t = \mathcal{S}pec\, B$ for an $A/\mathfrak{m}$-algebra B and $\dim_{A/\mathfrak{m}} B = n$ as a vector space.

DEFINITION 13. An affine morphism $X \to Y$ is flat/finite if for any cartesian diagram

$$\begin{array}{ccc} \mathcal{S}pec\, B & \longrightarrow & \mathcal{S}pec\, A \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

B is flat/finite A-algebra. Note that being a finite A-algebra means being finitely generated as an A-module.

EXAMPLE 14. $X = \mathbb{A}^1$ and $A = k$ a field. Then any $Z = \mathcal{S}pec\, k[t]/I$ is flat over k obviously, and is finite since $k[t]$ is a PID and hence $I = (f)$ for some polynomial f, thus $k[t]/(f)$ is finite dimensional.

$Hilb^n(\mathbb{A}^1)(k) = \{$ closed subscheme $Z \subset \mathbb{A}^1_k$ eq. $I \subset k[t]$ such that $Z = \mathcal{S}pec(k[t]/I)$ has our properties$\}$

It remains to note that we want the fibers (here being only $\mathcal{S}pec\, k[t]/I = \mathcal{S}pec\, k[t]/(f)$) be n-dimensional k-vector space, so we demand f to be a degree n polynomial. So we showed that

$$Hilb^n(\mathbb{A}^1)(k) \cong k^n$$

in a canonical way.

We will prove now the following

PROPOSITION 3.

$$Hilb^n(\mathbb{A}^1) = \mathbb{A}^n$$

.

PROOF. It suffices to show that $Hilb^n(\mathbb{A}^1)(A) = \mathbb{A}^n(A)$ in a natural way! We are looking for ideals $I \subset A[t]$ such that $B = A[t]/I$ is flat and finitely generated as A-module and for every maximal ideal $\mathfrak{m} \subset A$, we have that $B/\mathfrak{m}B = B \otimes_A A/\mathfrak{m}$ is a vector space of dimension n over $A/\mathfrak{m}$. By proposition 2 this corresponds precisely to ideals $I \subset A[t]$ such that $B = A[t]/I$ is Zariski locally free of rank n.

Claim. If I is Zariski locally free of rank n over A, then $I = (f)$ for a unique monic $f \in A[t]$.

Consider $1, t, \cdots, t^{n-1} \in B$; these give a basis of $B/\mathfrak{B}$ over $A/\mathfrak{A}$ for any maximal $\mathfrak{m} \subset A$ as $B/\mathfrak{B} \cong A[t]/\mathfrak{m}A[t] \cong A/\mathfrak{m}[t]$. ?

□

Note 7.1. If k is an algebraically closed field then for such f we have the splitting $f = \prod_{i=1}^n (t - \alpha_i)$ hence any $Z \subset \mathbb{A}^1_k$ corresponds to the finite set of points $\alpha_1, \cdots, \alpha_n \in \mathbb{A}^1_k$. But we always a functor in the other direction:

$$\mathbb{A}^n \to Hilb^n(\mathbb{A}^1)$$
$$(\alpha_1, \cdots, \alpha_n) \mapsto \prod_{i=1}^n (t - \alpha_i)$$

So we can think of $Hilb^n(\mathbb{A}^1)$ as the quotient of $\mathbb{A}^n$ by S_n as the permutation of α_i's is irrelevant. ⌟

REMARK. $Hilb^n(\mathbb{P}^1) = \mathbb{P}^n$.

8. Sheaves/Schemes

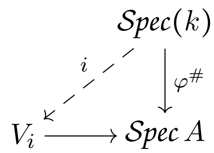
Recall that a presheaf is just a functor $X : R - alg \to Set$.

DEFINITION 14. (1) Let $\{U_i \to X\}_{i \in I}$ be a family of open immersions of presheaves. The family is a (Zariski) open covering if for any A and for all $x \in X(A)$ when we complete to cartesian diagrams

$$\begin{array}{ccc} V_i & \longrightarrow & \mathcal{S}pec\, A \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & X \end{array}$$

the fibered products V_i satisfy $\{V_i \to \mathcal{S}pec\, A\}$ is a Zariski open covering of $\mathcal{S}pec\, A$ in the following sense!

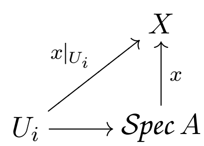
(2) $\{V_i \to \mathcal{S}pec\,A\}$ is an open covering if for any field k and all R-algebra morphism $A \xrightarrow{\varphi} k$ there exists i such that $\varphi^{\#}$ factors through V_i making the following diagram commute



Definition 15. The presheaf X is a sheaf if for all R-algebra A, and for all open coverings $\{U_i \to \mathcal{S}pec\,A\}_{i\in I}$, the following sequence is exact.

$$0 \to X(A) \to \prod X(U_i) \rightrightarrows \prod_{i,j} X(U_{ij})$$

here we have more generally than ever defined $X(U) = \mathrm{Mor}_{presh}(U, X)$. The last maps are easy to describe on the image objects, the first one $X(A) \to \prod X(U_i)$ is given vie $x : X \to \mathcal{S}pec\,A \mapsto x|_{U_i}$



Remark. (1) These are Zariski sheaves; we can use different notions of coverings and get different notions of sheaves.

(2) Let X be a Zariski sheaf. Then we get an induced sheaf $X|_{\mathcal{S}pec\,A}$ on the topological space $\mathcal{S}pec\,A$, for every A.

$$X|_{\mathcal{S}pec(A)}(U) = \mathrm{Mor}_{presh}(U, X)$$

where $U \subset \mathcal{S}pec\,A$ is Zariski open subset.

(3) X is a "big sheaf" and $X|_{\mathcal{S}pec\,A}$ is a "small sheaf".

Lemma 5. *For X to be a sheaf it suffices to check that for any A and $a_1, \cdots, a_n \in A$ such that $\langle a_1, \cdots, a_n\rangle = A$ then following is exact*

$$0 \to X(A) \to \prod_{i=1}^{n} X(\mathcal{S}pec\,A_{a_i}) \rightrightarrows \prod_{i,j} X(\mathcal{S}pec\,A_{a_i} \cap \mathcal{S}pec\,A_{a_j})$$

Proof. This follows from sheaf theory on any space and $\mathcal{S}pec\,A_f \subset \mathcal{S}pec\,A$ form a basis of topology. □

Lemma 6. *Let B be an R-algebra, then $\mathcal{S}pec\,B$ is a sheaf.*

Proof. From the previous lemma we need only to take some R-algebra A, where $\langle a_1, \cdots, a_n\rangle = A$ and show exactness of

$$\mathrm{Hom}(B, A) \to \prod \mathrm{Hom}(B, A_{a_i}) \rightrightarrows \mathrm{Hom}(B, A_{a_i a_j})$$

For the first arrow suppose $\varphi, \psi : B \to A$ and the compositions $\varphi_i, \psi_i : B \xrightarrow{\varphi,\psi} A \hookrightarrow A_{a_i}$ are identical for all i. Thus $\psi_i(b) - \varphi_i(b) = 0 \in A_{a_i}$ for all $b \in B$ and for any i. So $\psi(b) - \varphi(b) \mapsto 0$ in $\prod A_{a_i}$. But $A \to \prod_i A_{a_i}$ is injective hence $\varphi = \psi$. In fact in the same paradigm, the exactness follows from the exactness of the following diagram

$$0 \to A \to \prod_i A_{a_i} \rightrightarrows \prod_{i,j} A_{a_i a_j}.$$

□

Definition 16. A presheaf X is a scheme if

(1) X is a sheaf, and
(2) There exists an open covering $\{U_i \to X\}$ of X such that each U_i is an affine scheme. ?

Remark. Let $(S, \mathcal{O}_S)$ be a scheme in the usual sense. This induces a functor $X : R-alg \to Set$ via $A \mapsto \mathrm{Mor}_{sch}(\mathcal{S}pec\, A, S)$. We claim that X satisfies the new definition of scheme.

Example 15. $\mathbb{P}^n$ is a scheme. For any R-algebra, A, such that $\langle a_1, \cdots, a_k \rangle = A$,

$$0 \to \mathbb{P}^n(A) \to \prod \mathbb{P}^n(A_{a_i}) \rightrightarrows \prod \mathbb{P}^n(A_{a_i a_j})$$

should be shown to be exact. Given $(L_i, \ell_{0_i}, \cdots, \ell_{n_i})$ a collection of descent data for $i = 1, \cdots, n$, such that we have isomorphisms

$$\varphi_{ij} : (L_i)_{a_i a_j} \to (L_j)_{a_j a_i}$$

sending $\ell_{k_i} \mapsto \ell_{k_j}$ for all k_i's, we need to construct an invertible A-module L with sections $\ell_0, \cdots, \ell_n$ an isomorphisms $\varphi : L_{a_i} \to L_i$ mapping $\ell_j \mapsto \ell_{j_i}$. Thus we have

$$\prod_i L_i \rightrightarrows \prod_{i,j} (L_i)_{a_i a_j}$$
$$(x_i)_i \rightrightarrows (x_i)_{i,j}, \varphi_{ij}^{-1}((x_j)_{j,i})$$

Define

$$L = \{x = (x_i)_i \in \prod L_i : \alpha(x) = \beta(x)\}$$

This has the structure of an A-module because of A-linearity of α and β. Now we want to check that L is invertible: suffices to prove

$$L_{a_i} \cong L_i$$

since invertibility is a local property. This follows from the isomorphism $\varphi_i : L_{a_i} \to L_i$ which we will define promptly. Let's first define the sections of L. Let $\ell_k = (\ell_{k_i})_i$ which is a well-defined element of L because of the definition of φ_{ij}. We also demand that $(\ell_0, \cdots, \ell_n)$ generate L. Construction:

$$\begin{array}{ccc} L_{a_i} & \overset{\varphi_i}{\dashrightarrow} & L_i \\ \uparrow & \nearrow_{\text{proj}} & \\ L & & \end{array}$$

Since L_i is an A_{a_i}-module get a unique map: $L \otimes_A A_{a_i} \to L_i$ via $\ell \otimes a \mapsto a.\text{proj}$. Verify locally that φ_i is an isomorphism. Without loss of generality now suppose L_i is free of rank 1. The remaining claims are local in $\mathcal{S}pec\, A$ and can be checked in a Zariski neighborhood of any maximal ideal $\mathfrak{m}$ of A. Thus without loss of generality let $L_i = A$, then

$$L = \ker\Big(\prod A_{a_i} \xrightarrow{\alpha-\beta} \prod A_{a_i a_j}\Big) = A\checkmark$$

So $\mathbb{P}^n$ is a sheaf!

Next we show $\mathbb{P}^n$ is locally affine. Let $U_i : \mathbb{A}^n \xrightarrow{\varphi_i} \mathbb{P}^n$ via $(a_1, \cdots, a_n) \mapsto (A, a_1, \cdots, 1, \cdots, a_n)$ defines an open immersion and $\{U_i \to \mathbb{P}^n\}_{i=0,\cdots,n}$ is an affine Zariski open cover. It suffices to show the following: Given a field k and element $\langle a_0, \cdots, a_n\rangle \in \mathbb{P}^n(k) \cong k^{n+1} - \{0\}/k^*$, we can find a_i non-zero such that

$$\langle a_0, \cdots, a_n\rangle = \langle \frac{a_0}{a_i}, \cdots, 1, \cdots, \frac{a_n}{a_i}\rangle.$$

So $\mathcal{S}pec(k)$ factors through U_i and this completes the proof.

Definition 17. A scheme X is quasi-projective if there exists an open and closed subscheme $U, Z \subset \mathbb{P}^n$ such that $X = U \cap Z$.

Proposition 4. *If X is quasi-projective then $Hilb^n(X)$ is a (quasi-projective) scheme.*

Proof. First we show that $Hilb^n(X)$ is a sheaf. We know that $Hilb^n(X)(A)$ is the set of closed immersions $Z \hookrightarrow X \times \mathcal{S}pec\, A$ such that $Z \to \mathcal{S}pec\, A$ is finite, affine, flat and the fibers are all of length n. (Note: We say an affine scheme $\mathcal{S}pec\, B$ over field k has length n if $\dim_k B = n$.) We want to consider algebras $A = \langle a_1, \cdots, a_n\rangle$. Suppose that we have the local data that

$$\begin{array}{ccc} Z_i & \longrightarrow & X \times \mathcal{S}pec\, A_{a_i} \\ & \searrow & \downarrow \\ & & \mathcal{S}pec\, A_{a_i} \end{array}$$

such that for any i, j we have an isomorphism $Z_i|_{\mathcal{S}pec\, A_{a_i a_j}} \overset{\varphi_{ij}}{\to} Z_j|_{\mathcal{S}pec\, A_{a_j a_j}}$ and want to construct a global affine $Z = \mathcal{S}pec\, B$.

$$\begin{array}{ccc} Z & \longrightarrow & X \times \mathcal{S}pec\, A \\ & \searrow & \downarrow \\ & & \mathcal{S}pec\, A \end{array}$$

The idea is to show that $Z = \mathcal{S}pec\, B$ where B is constructed from the local data, as a locally free A-module. To construct it locally we use sheaf properties, then we check that it is a closed immersion locally! □ ?

Define $\mathfrak{Z} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^n$ as the zero-locus of $t_0 x^n + t_1 x^{n-1} y + \cdots + t_n y^n$, where t_i's are coordinates on $\mathbb{P}^n$ and x, y are coordinates on $\mathbb{P}^1$. This polynomial is homogeneous of degree 1 in t and of degree n in x and y. The claim is that $\mathfrak{Z} \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^n \to \mathbb{P}^n$ is affine, finite, flat and the fibers are of length n. So $\mathfrak{Z}$ defines a morphism $\mathbb{P}^n \to Hilb^n(\mathbb{P}^1)$. We now want to check that this is an isomorphism. We only need to check this locally as both are sheaves. This completes the proof that

PROPOSITION 5. *$Hilb^n(\mathbb{P}^1) = \mathbb{P}^n$.*

The last thing to prove in this section is that

PROPOSITION 6. *$Hilb^\ell(\mathbb{A}^n)$ is a quasi-projective scheme.*

9. Grassmannians

The presheaf $G(m,n)$ is defined by

$$\begin{aligned} G(m,n)(A) = \{ & \text{ short exact sequences of } A\text{-modules } 0 \to U \to A^n \to Q \to 0 \\ & \text{such that } U \text{ and } Q \text{ are finite, flat of degree } m \text{ and } n-m \text{ respectively } \}/\cong \end{aligned}$$

The equivalence relation on the short exact sequences is existence of the following commutative diagrams:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & A^n & \longrightarrow & Q & \longrightarrow & 0 \\ & & \Downarrow & & \| & & \Downarrow & & \\ 0 & \longrightarrow & U' & \longrightarrow & A^n & \longrightarrow & Q' & \longrightarrow & 0 \end{array}$$

Note that a quotient Q as above (finite flat), determines the finite, flat, degree $n-m$ kernel U in other words:

$$\begin{aligned} G(m,n)(A) &= \{ \text{ submodules of } A^n \text{ such that the quotient is finite flat degree } n-m \ \} \\ &= \{ \text{ s.e.s } 0 \to U \to O^n_{\mathcal{S}pec\, A} \to Q \to 0 \text{ where } U, Q \\ &\qquad \text{are locally free coherent of ranks } m \text{ and } n-m \ \} \end{aligned}$$

PROPOSITION 7. *$G(m,n)$ is a projective scheme.*

PROOF.

$$G(m,n) \xrightarrow{\rho} \mathbb{P}^{\binom{n}{n-m}-1}$$

$$(A^n \twoheadrightarrow Q) \mapsto \Big(\bigwedge_A^{n-m} A^n \twoheadrightarrow \bigwedge_A^{n-m} Q\Big)$$

The canonical basis of $\bigwedge^{n-m} A^n$ gives $\binom{n}{n-m}$ elements $\ell_i \in \bigwedge_A^{n-m} Q$ so $(\bigwedge_A^{n-m} Q, \ell_0, \cdots, \ell_{\binom{n}{n-m}-1}) \in \mathbb{P}^{\binom{n}{n-m}-1}$. Note that $\bigwedge^{n-m} A^n$ is again free.

Claim: ρ is a closed immersion. For this it suffices to show that ρ_0 is a closed immersion for all standard open affines of $\mathbb{P}^{\binom{n}{m}-1}$.

$$\begin{array}{ccc} \mathbb{A}^{m(n-m)} & \xrightarrow{\rho_0} & \mathbb{A}^{\binom{n}{m}-1} \\ \downarrow & & \downarrow i_0 \\ G(m,n) & \xrightarrow{\rho} & \mathbb{P}^{\binom{n}{m}-1} \end{array}$$

Tracing on an R-algebra A we have

$$\begin{array}{ccc} \mathrm{Hom}(A^m, A^{n-m}) & \xrightarrow{\rho_0} & A^{\binom{n}{m}-1} \\ \downarrow & & \downarrow \\ G(m,n)(A) & \xrightarrow{\rho} & \mathbb{P}^{\binom{n}{m}-1}(A) \end{array}$$

The left downwards arrow is $\varphi \in M_{n-m\times m}(A) \mapsto (I_{n-m\times n-m}|\varphi) : A^n \twoheadrightarrow A^{n-m}$. The upper right arrow is

$$\varphi \mapsto \text{ all determinants of } n-m\times n-m \text{ minors of } (I|\varphi)\ .$$

It is easy to check that the diagram is cartesian. It remains to prove that ρ_0 is a closed immersion. This is equivalent to having that the corresponding maps of rings is a surjection. This follows of existence of π such that $\pi \circ \rho_0 = \mathrm{id}$. The determinant of minors of $(I|\varphi)$ gives φ, and this is the mapping π. □

Now: $Hilb^{\ell}(\mathbb{A}^n)$ is a locally closed subscheme of some $G(M,N)$. For the general case of the proof that $Hilb^p(X)$ is a quasi-projective scheme look at [Bertram]. Here we give a proof of this specific case along the same ideas.

TO DO

10. Properties of the Hilbert schemes of n points

Up to here we have shown that if X is a quasi-projective scheme over R then $X^{[n]} := Hilb^n X$ is a quasi-projective scheme by showing that it is a locally closed subscheme of a Grassmannian. We have seen that if k is a field, then $X^{[1]} = X$, $Hilb^n(\mathbb{A}^1) = \mathbb{A}^n$. Now we will show that $Hilb^n(\mathbb{A}^2)$ is a smooth variety if R is a field k.

We would like to write $Hilb^n(\mathbb{A}^r) = \widetilde{H}^n/\mathrm{Gl}_n$ where $\widetilde{H}^n$ is a scheme and Gl_n acts freely on it. We will then show that in the case when $r = 2$, $\widetilde{H}^n$ is smooth.

Proposition 8.

$$\begin{aligned} Hilb^n(\mathbb{A}^2) = \{&(V, B_1, B_2, v_0) :\ V \textit{ is an } n\textit{-dimensional vector space, } B_i \in \mathrm{End}(V) \\ &\textit{and } [B_1, B_2] = 0 \textit{ and } v_0 \in V \textit{ such that there is no } W \subsetneq V \textit{ containing } v_0 \textit{ and } B_1, B_2 \textit{ are stable}\}/\cong \end{aligned}$$

The rest of this section is the proof of this theorem. More precisely we will show

$$\begin{aligned} &Hilb^n(\mathbb{A}^2)(A) = \{(V, B_1, B_2, v_0) :\ V \text{ is a finite flat } A\text{-module of rank } n,\ B_i \in \operatorname{End}_A(V) \qquad (10.1)\\ &\text{and } [B_1, B_2] = 0 \text{ and } v_0 \in V \text{ such that } A[x_1, x_2] \to V \text{ via } x_i \mapsto B_i(v_0)\}/\cong \qquad (10.2)\end{aligned}$$

By definition $Hilb^n(\mathbb{A}^2)(A)$ is the set of commutative diagrams

$$\begin{array}{ccc} B & \overset{\varphi}{\twoheadleftarrow} & A[x_1, x_2] \\ & \nwarrow & \uparrow \\ & & A \end{array}$$

where B is a finite flat A-algebra of rank n. Associat to this the following data:

$$V = B, \quad B_i = \text{multiplication by } \varphi(x_i), \quad v_0 = \varphi(1) = 1_B$$

Note that under this, $[B_1, B_2] = 0$ and under $\varphi : A[x_1, x_2] \to B$ we have $x_i \mapsto \varphi(x_i).1 = B(v_0)$. Moreover,

$$f(x_1, x_2) \mapsto f(B_1, B_2)(v_0)$$

and is surjective. Thus the properties in 10.1 holds here.

Conversely given data as in 10.1,

$$I = \{f(x_1, x_2) \in A[x_1, x_2] : f(B_1, B_2) = 0 \in \operatorname{End}(V)\}.$$

To show that $0 \to I \to A[x_1, x_2] \to V \to 0$ is exact where the last mapping is given by $f(x_1, x_2) \mapsto f(B_1, B_2)(v_0)$, beside trivial things, we have to show that $f(B_1, B_2)(v_0) = 0$ implies $f(B_1, B_2) = 0$. Let $v \in V$ be arbitrary vector, given by $v = g(B_1, B_2)(v_0)$, then

$$f(B_1, B_2)v = f(B_1, B_2).g(B_1, B_2).v_0 = 0.$$

This completes the proof that $V \cong A[x_1, x_2]/I$ as A-modules. Now we want to endow V with a ring structure to get that $\varphi : A[x_1, x_2] \to V$ is an A-algebra morphism. We define the multiplication by

$$(f(B_1, B_2)v_0).(g(B_1, B_2)v_0) = f(B_1, B_2)g(B_1, B_2)v_0.$$

Now define

$$\widetilde{H}^n(A) = \{(B_1, B_2, v_0) \in M_{n\times n}(A) \times M_{n\times n}(A) \times A^n, \text{ such that } [B_1, B_2] = 0, A[x_1, x_2] \twoheadrightarrow A^n\}$$

which is a subfunctor of $M_{n\times n} \times M_{n\times n} \times \mathbb{A}^n \cong \mathbb{A}^{2n^2+n}$.

The condition $[B_1, B_2] = 0$ defines a closed subscheme of $\mathbb{A}^{2n^2+n}$. The surjectivity of $A[x_1, x_2] \twoheadrightarrow A^n$ translates to the fact that if we associate to $A[x_1, x_2]$ the corresponding quasi-coherent sheaf on the affine space, and to A^n the coherent sheaf. The cokernel which is hence coherent is supported away from our desired points! The support of a coherent sheaf is closed hence this is an open condition. So $\widetilde{H}^n$ is an open subscheme of a closed subscheme of $\mathbb{A}^{2n^2+n}$.

Define the morphism of schemes $\widetilde{H}^n \overset{\pi}{\to} Hilb^n(\mathbb{A}^n)$ via

$$(B_1, B_2, v_0) \mapsto (A^n, B_1, B_2, v_0).$$

Next step is to involve the action of Gl_n. Define an action

$$\begin{aligned} \mathrm{Gl}_n \times \widetilde{H}^n &\overset{sigma}{\to} \widetilde{H}^n \\ \mathrm{Gl}_n(A) \times \widetilde{H}^n(A) &\to \widetilde{H}^n(A) \\ g.(B_1, B_2, v_0) &\mapsto (gB_1g^{-1}, gB_2g^{-1}, gv_0) \end{aligned}$$

Lemma 7. *We have a cartesian diagram:*

$$\begin{array}{ccc} \mathrm{Gl}_n \times \widetilde{H}^n & \xrightarrow{\sigma} & \widetilde{H}^n \\ {\scriptstyle \mathrm{proj}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \widetilde{H}^n & \xrightarrow[\pi]{} & Hilb^n(\mathbb{A}^2) \end{array}$$

Proof. The proof of this is easy; remember that one step of it is to show that given $(B_1, B_2, v_0), (B_1', B_2', v_0') \in \widetilde{H}^n(A)$ there is an isomorphism $g : A^n \to A^n$ in $g \in \mathrm{Gl}_n(A)$ sych that $g.(B_1, B_2, v_0) = (B_1', B_2', v_0')$. □

Remark. This g is unique; we know g on B_1v_0 and B_2v_0 and this determines g on all of A^n.

Lemma 8. $\pi : \widetilde{H}^n \to Hilb^n(\mathbb{A}^2)$ *is sheaf surjective.*

Proof. This means that for all R-algebra A and all mappings $\mathcal{S}pec\, A \to Hilb^n(\mathbb{A}^2)$ there is an open covering of $\mathcal{S}pec\, A$, $\langle f_1, \cdots, f_n \rangle = 1$ such that for all $i \in \{1, \cdots, s\}$ there is a dotted map making the following diagram commutes:

$$\begin{array}{ccccc} & & & \nearrow & \widetilde{H}^n \\ & & & & \downarrow \\ \mathcal{S}pec(A_{f_i}) & \hookrightarrow & \mathcal{S}pec\, A & \longrightarrow & Hilb^n(\mathbb{A}^2) \end{array}$$

Take a covering over which V becomes free. Then over $\mathcal{S}pec\, A_{f_i}$ choose isomorphism $A^n_{f_i} \cong V_{f_i}$, via this isomorphism B_1, B_2 map to matrices in $M_{n\times n}(A_{f_i})$ which we take as B_1' and B_2'. And v_0 maps to an element of A_{f_i} which we take as v_0'. This defines a lift $\mathcal{S}pec(A_{f_i}) \to \widetilde{H}^n$. □

Remark. The above lemmas say that $\widetilde{H}^n \to Hilb^n(\mathbb{A}^2)$ is a principle bundle with structure group GL_n; i.e. a GL_n-bundle. This definition makes sense for X is a scheme and Y a GL-scheme and $Y \xrightarrow{\pi} X$ a GL_n-invariant maps. For other groups the Zariski topology on X may not be fine enough, and we must use the etale topology instead. Zariski topology however works for GL_n, SL_n and Sp_{2n}. ?

We conclude that

COROLLARY 3. *There exists a Zariski open covering* $\{U_i\}$ *of* $Hilb^n(\mathbb{A}^2)$ *such that the following is cartesian:*

$$\begin{array}{ccc} \mathrm{Gl}_n \times U_i & \xrightarrow{\text{open}} & \widetilde{H}^n \\ \downarrow{\scriptstyle \text{proj}} & & \downarrow{\scriptstyle \pi} \\ U_i & \xrightarrow[\text{open}]{} & Hilb^n(\mathbb{A}^2) \end{array}$$

PROOF. For this note that $Hilb^n(\mathbb{A}^2)$ is a scheme. So it has an affine covering. Refine this covering to get another one by the previous lemma which has lifts

$$\begin{array}{ccc} & & \widetilde{H}^n \\ & \nearrow^{s_i} & \downarrow \\ U_i & \hookrightarrow & Hilb^n(\mathbb{A}^2) \end{array}$$

We get cartesian squares:

$$\begin{array}{ccccc} \mathrm{GL}_n \times U_i & \longrightarrow & \mathrm{GL}_n \times \widetilde{H}^n & \longrightarrow & \widetilde{H}^n \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \sigma} \\ U_i & \xrightarrow{s_i} & \widetilde{H}^n & \xrightarrow{\pi} & Hilb^n(\mathbb{A}^2) \end{array}$$

Since $\pi \circ s_i$ is the inclusion it is an open immersion, implying that the top composition is also an open immersion. □

The trivial principal GL-bundle over X is $\mathrm{GL}_n \times X \to X$ with GL_n action by left multiplication. Every GL_n-bundle is locally trivial by definition.

REMARK. All the above works with $\mathbb{A}^r$ not just $\mathbb{A}^2$, get principal GL_n-bundle

$$\widetilde{H}^n \to Hilb^n(\mathbb{A}^r)$$

where

$$\widetilde{H}^n = \{(B_1, \cdots, B_n, v_0) \in M^r_{n\times n} \times A^n : \forall i, j [B_i, B_j] = 0, A^n \text{ generated by } B_i(v_0)\text{s}\}.$$

However the following depends heavily on $r = 2$:

THEOREM 10.1. *Let* $R = k$ *an algebraic closed field,* $\mathrm{char}(k) = 0$ *and consider* $Hilb^n(\mathbb{A}^2)$. *Then* $\widetilde{H}^n$ *is a smooth subvariety of* $M^2_{n\times n} \times \mathbb{A}^n_k$.

Here on, we assume $R = k$ is algebraically closed.

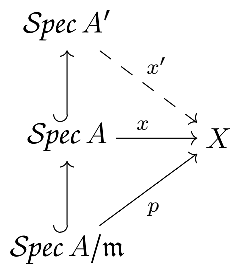
DEFINITION 18. A small extension is a pair $I \leq A^1$ where A^1 is a finite dimensional local k-algebra and I is an ideal of A^1 with dimension one over k.

Let $A = A^1/I$. Then from the exact sequence $0 \to I \to A^1 \to A \to 0$ we have that A^1 is an extension of A by I. Let $\mathfrak{m}$ be the maximal ideal of A and $\mathfrak{m}^1$ be the maximal ideal of A^1. Then $\mathfrak{m}^1$ is a flat A^1 module and hence $\mathfrak{m}^1 \otimes_{A^1} I \cong I\mathfrak{m}^1 = 0$ hence $I^2 = 0$. However I is an A^1-module and since $I.I = 0$ we get a well-defined action of A on I. Moreover this action factors to give an action of $A/\mathfrak{m}$ on I. So $I \cong A/\mathfrak{m}$ as a vector space over $A/\mathfrak{m}$. ?

Example 16. Small curvilinear extension: $A^1 = k[t]/(t^{\ell+1})$, $A = k[t]/t^\ell$ and $I = \langle t^\ell \rangle$.

Definition 19. A k-presheaf is smooth if for every small extension $I \to A' \to A$, $X(A') \to X(A)$ is surjective.

If $p \in X(k)$ is a point, X is smooth at p if considering $X(A) \to X(A/\mathfrak{m}) = X(k)$ for any $x \in X(A)$ that maps to $p \in X(k)$ under $X(A) \to X(A/\mathfrak{m})$ then there is $x' \in X(A')$ such that $x' \mapsto x$.



Example 17. Take $A' = k[t]/(t^3)$ and $A = k[t]/(t^2)$ and $I = \langle t^2 \rangle$. Take $X = \mathcal{S}pec\, k[x,y]/(y^2 - x^3)$. THen $X(A') \to X(A)$ is not surjective; take $(t,t) \in X(k[t]/(t^2))$. Then (t,t) is living above the point $p : \mathcal{S}pec\, A\mathfrak{m} \to X$ given by the origin, and hence X is not smooth at the origin.

Suppose char$k \neq 2$.

Proposition 9. *$\widetilde{H}$ is smooth.*

Lemma 9. *Consider the trace form for a small extension $I \to A' \to A$ given by*

$$\begin{aligned} M_{n\times n}(A') \times M_{n\times n}(I) &\to I \\ (\xi, X) &\mapsto \operatorname{tr}(\xi X). \end{aligned}$$

If $W \subset M_{n\times n}(I)$ is an A-submodule, then $W \to W^{\perp\perp}$ is sujrective.

Proof. The trace form is a non-degenrate symmetric bilinear pairing and we may think of $W \subset M_{n\times n}(k) \cong k^{n^2}$ in lieu of the identification $I \cong k$. □

Choose $(B_1, B_2, v_0) \in \widetilde{H}^n(A)$.

Lemma 10. *Let $W = \{[B_1, X_2] - [B_2, X_1] \in M_{n\times n}(I) : X_1, X_2 \in M_{n\times n}(I)\}$. Then*

$$W^\perp = \{\xi \in M_{n\times n}(A') : \exists f \in A[z_1, z_2], \xi = f(B_1, B_2) \mod \mathfrak{m}'\}.$$

TO DO

Lemma 11. *Let $\widetilde{B}_1, \widetilde{B}_2$ be arbitrary lefts of B_1, B_2 to $M_{n\times n}(A')$. Then $[\widetilde{B}_1, \widetilde{B}_2] \in M_{n\times n}(I)$ and is in $(W^{\perp})^{\perp}$.*

TO DO

Let $(\widetilde{B}_1, \widetilde{B}_2, \widetilde{v}_0)$ be any lift of (B_1, B_2, v_0) to A'. Then $[\widetilde{B}_1, \widetilde{B}_2] \in W$. So there are $X_1, X_2 \in M_{n\times n}(I)$ such that

$$[\widetilde{B}_1, \widetilde{B}_2] = [B_1, X_2] - [B_2, X_1]$$

Thus $[\widetilde{B}_1 - X_1, \widetilde{B}_2 - X_2] = 0$ (since $[X_1, X_2] \in M_{n\times n}(I^2)$ and $I^2 = 0$). On the other hand,

$$\begin{array}{ccc} A'[z_1,z_2] & \longrightarrow & A'^n \\ \Big\downarrow & & \Big\downarrow \\ A[z_1,z_2] & \twoheadrightarrow & A^n \end{array}$$

The lower arrow is surjective and by Nakayama's lemma for the A'-module A'^n we have that the above arrow is also surjective (lift generators). This completes the proof of the fact that

$$(\widetilde{B}_1 - X_1, \widetilde{B}_2 - X_2, \widetilde{v}_0) \in \widetilde{H}(A')$$

and that consquently $\widetilde{H}^n$ is smooth.

11. Tangent space to $\widetilde{H}$

Let $p = (B_1, B_2, v_0) \in \widetilde{H}(k)$. The tangent space of $\widetilde{H}$ at p is given by

$$T_p\widetilde{H} = \text{ preimage of } p \text{ under } \widetilde{H}(k[\varepsilon]) \to \widetilde{H}(k).$$

This preimage consists of all $(B_1 + \varepsilon X_1, B_2 + \varepsilon X_2, v_0 + \varepsilon x_0)$ such that $X_1, X_2 \in M_{n\times n}(I)$ and $x_0 \in k^n$ satisfying $[B_1 + \varepsilon X_1, B_2 + \varepsilon X_2] = 0$, i.e.

$$T_p\widetilde{H} = \{(X_1, X_2, x_0) : [B_1, X_2] = [B_2, X_1]\}.$$

For $A' = k[\varepsilon], A = k, I = \langle\varepsilon\rangle$, we replace the two-form $\mathrm{tr} : M_{n\times n}(k[\varepsilon]) \times M_{n\times n}(\varepsilon) \to \langle\varepsilon\rangle$ by

$$\mathrm{tr}(.,.) : M_{n\times n}(k) \times M_{n\times n}(k) \to k$$

Then we have $W \subset M_{n\times n}(k)$ with the notation as above, $k[z_1, z_2] \to W^{\perp}$ via $f(z_1, z_2) \mapsto f(B_1, B_2)$ is surjective with kernel

$$K = \{f \in k[z_1, z_2] : f(B_1, B_2)v_0 = 0\}.$$

but under the mapping $f(z_1, z_2) \mapsto f(B_1, B_2)v_0$ we have the short exact sequence

$$0 \to K \to k[z_1, z_2] \to k^n \to 0$$

hence $W^{\perp} \cong k^n$, so $\dim(W^{\perp}) = n$ and hence $\dim(W) = n^2 - n$.

On the other hand W fits into the short exact sequence

$$0 \to T_p\widetilde{H} \to M_{n\times n}(k)^2 \times k^n \to W \to 0$$

where the projection is given by $(X_1, X_2, X_0) \mapsto [B_1, X_2] - [B_2, X_1]$. Hence

$$\dim T_p\widetilde{H} = 2n^2 + n - n^2 + n = n^2 + 2n$$

and thus $\dim \widetilde{H} = n^2 + 2n$ and $\dim(\widetilde{H}/\mathrm{GL}_n) = 2n$ and finally

COROLLARY 4. *$Hilb^n(\mathbb{A}^2)$ is a smooth scheme of dimension $2n$.*

12. Points of $Hilb^n(\mathbb{A}^s)$ as locus of n points in $\mathbb{A}^s$

DEFINITION 20. The point $[B_1, \cdots, B_s, v_0] \in Hilb^n\mathbb{A}^s$ is semi-simple if all B_i's are diagonalizable. Since they commute, they are simultaneously diagonalizable, so we have that such a point is equivalent to $[D_1, \cdots, D_s, v_0']$.

Note that all entries of v_0' are non-zero from the spanning criterion. Let $D_i = \begin{pmatrix} \lambda_1^{(i)} & & 0 \\ & \ddots & \\ 0 & & \lambda_n^{(i)} \end{pmatrix}$, and define $x_j = (\lambda_j^{(1)}, \cdots, \lambda_j^{(n)}) \in \mathbb{A}^s$. Then $x_1, \cdots, x_n \in \mathbb{A}^s$ are n distinct points (again by spanning criterion). The converse procedure from n points $x_1, \cdots, x_0 \in \mathbb{A}^s$ to getting points on Hilbert scheme, is obvious. So

$$Hilb^n_{ss}(\mathbb{A}^s) = (\mathbb{A}^s)^n_0.$$

?

?

For simplicity work with $s = 2$. We wil work on an algebraically closed field k. If $[B_1, B_2, v_0] \in Hilb^n(\mathbb{A}^2)(k)$ we get data $(V_i, \lambda_i, \mu_i)_{i\in\{1,\cdots,r\}}$ such that $V = \oplus_{i=1}^r V_i$ into generalized eigenspaces. Assume that $\dim V_i \geq \dim V_{i+1}$ so that we get a partition of n given by $(\alpha_1, \cdots, \alpha_r)$. This partition corepsonds to $[B_1, B_2, v] \in Hilb^n(\mathbb{A}^2)$. The subscheme of $\mathbb{A}^2$ correpsonding to $[B_1, B_2, v]$ is the ideal

$$I = \{f \in k[z_1, z_2] : f(B_1, B_2) = 0 \in \mathrm{End}(k^n)\}$$

The polynomials $\prod_{i=1}^r (z_1 - \lambda_i)^N$ and $\prod_{i=1}^r (z_1 - \mu_i)^N$ are in the ideal and so is $\prod_I (z_1 - \lambda_i)^N \prod_J (z_2 - \mu_j)^N$ for any index sets $I \coprod J = \{1, \cdots, r\}$. Thus $(a, b) \in Z(I)$ iff $(a, b) = (\lambda_i, \mu_i)$ for some $i \in \{1, \cdots, r\}$.

13. Euler characteristics

Let X be a scheme over $\mathbb{C}$. Then $X(\mathbb{C})$ is a topological space. We want to compute $\chi(X(\mathbb{C}))$.

DEFINITION 21. The analytic topology:

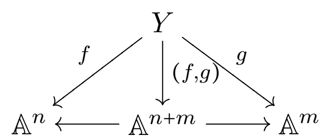
(1) $X(\mathbb{C})$ has the Zariski topology: $U \subset X(\mathbb{C})$ is open iff there is an open subscheme $X' \subset X$ such that $U = X'(\mathbb{C}) \hookrightarrow X(\mathbb{C})$.

(2) $X = \cup_i X_i$ open subschemes, $X_i \hookrightarrow \mathbb{A}^{n_i}$ closed subscheme $X(\mathbb{C}) = \cup_i X_i(\mathbb{C})$. And

$$X_i(\mathbb{C}) \hookrightarrow \mathbb{A}^{n_i}(\mathbb{C}) = \mathbb{C}^{n_i}$$

the analytic topology on $\mathbb{C}^{n_i}$ induces a topology on $X_i(\mathbb{C})$. Put the finest topology on $X(\mathbb{C})$ such that $X_i(\mathbb{C}) \hookrightarrow X(\mathbb{C})$ are all open subsets.

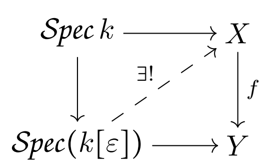
One would like to prove that the topology is independent of choices: the main point is that if $Y \hookrightarrow \mathbb{A}^n$ is a closed subscheme and likewise is a closed subscheme $Y \hookrightarrow \mathbb{A}^m$, then $Y(\mathbb{C})$ gets the same topology from $\mathbb{C}^n$ and $\mathbb{C}^m$.



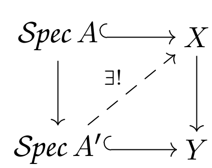
We may extend $Y \to \mathbb{A}^m$ to $G : \mathbb{A}^n \to \mathbb{A}^m$. Use continuity of G in the analytic topology.

Let $Y/\mathbb{C}$ be a smooth variety, $p \in Y(\mathbb{C})$, then there is an (analytic) open neighborhood such that $p \in U \subset Y(\mathbb{C})$, and $U \to \mathbb{C}^m$ is an analytic isomorphism onto open subset. Algebraically the best we can do is to take Y smooth and $p \in Y$ then there is a Zariski open neighborhood $p \in U \subset Y$ and $U \to \mathbb{A}^r$ is etale. Consider the example of elliptic curves for instance.

Definition 22. If X, Y are smooth then $f : X \to Y$ is etale if f is bijective on Zariski tangent spaces.



If X, Y are of finite type schemes over the algebraically closed field k, we will have then that for all small extensions $A' \to A$ the following diagram completes:



Remark. (1) If we only get the existence of the diagonal arrow, then f is smooth.

(2) If we only have the uniqueness of the diagonal arrow, then f is called unramified.

(3) If we put $Y = \mathcal{S}pec\, k$ the we get back the notion of smoothness (of X).

Let Y be a smooth variety over $\mathbb{C}$ and let $U \subset Y(\mathbb{C})$ be an open. We get open subsets of $Hilb^n Y(\mathbb{C})$ by considering all $[Z] \in Hilb^n Y(\mathbb{C})$ such that $Z \subset U \subset Y$. Denote this by $Hilb^n(U) \subset Hilb^n(Y)$.

REMARK. The map $Hilb^n(Y(\mathbb{C})) \to Y(\mathbb{C})^n/S_n$ is continuous. For $Y = \mathbb{A}^2$ we have that

$$Hilb^n \mathbb{A}^2(\mathbb{C}) \to (\mathbb{C}^2)^n/S_n$$

via $[B_1, B_2, v_0] \mapsto \sum_{i=1}^r \alpha_i(\lambda_i, \mu_i)$.

Then if $U \to \mathbb{A}^s$ is an isomorphism onto an open ste we get an induce map

$$Hilb^n(U) \to Hilb^n(\mathbb{A}^s)$$

from an open subset.[1] It is however not true that if U_i is a cover of Y then $Hilb^n(U_i)$ is a cover of $Hilb^n(Y)$.

Let $Z \subset Y$ be a subscheme of length n. Suppose $Z = Z_1 \cup \cdots \cup Z_r$ were these components are disjoint, and suppose there exists $U_i \supset Z_i$ with $U_i \hookrightarrow \mathbb{A}^s$ admitting holomorphic coordinates and for simplicity assume $U_i \cap U_j = \varnothing$. Let length Z_i be α_i. The partition of n being $\alpha = (\alpha_1, \cdots, \alpha_r)$, then the morphism

$$\prod_{i=1}^r Hilb^{\alpha_i} U_i \xrightarrow{\varphi} Hilb^n Y$$

via $(Z_1', \cdots, Z_r') \mapsto \cup_i Z_i'$ is an open neighborhood of $[Z]$. And via coordinates

$$\prod_{r=1}^r Hilb^{\alpha_i}(U_i) \hookrightarrow \prod_{i=1}^r Hilb^{\alpha_i}(\mathbb{A}^s)$$

is an open embedding. This is how we resolve the problem of finding an open cover of the Hilbert schemes of n points.

14. Computing the Euler characteristic of $Hilb^n(Y(\mathbb{C}))$

We do this when Y is a smooth algebraic variety over $\mathbb{C}$.

If X is a scheme over $\mathbb{C}$ then

$$\chi(X(\mathbb{C})) = \sum_{i \geq 0} (-1)^i \dim_{\mathbb{Q}} H_c^i(X(\mathbb{C}), \mathbb{Q})).$$

Recall that we know the

THEOREM 14.1. *If $U \subset X$ is an open subscheme, $Z \subset X$ is a closed subscheme such that the complete of Z is U then*

$$\chi(X(\mathbb{C})) = \chi(U(\mathbb{C})) + \chi(Z(\mathbb{C})).$$

PROOF. Use long exact sequence of cohomology (possibly with compact support). □

[1] Authors: $Hilb^n(U)$ is the pre-image of U^n/S_n along the Hilbert-Chow map

DEFINITION 23. A stratification of a scheme X is

(1) A finite index set I, partially ordered

(2) $\forall i \in I$ a locally closed subscheme $Z_i \subset X$ such that $X = \coprod_i Z_i$ and $\forall i \in I$

$$\bigcup_{j \geq i} Z_j \hookrightarrow X \text{ is closed and } Z_i \hookrightarrow \bigcup_{j \geq i} Z_j \text{ is open.}$$

Given a stratification (Z_i) of a scheme X, then

$$\chi(X(\mathbb{C})) = \sum_{i \in I} \chi(Z_i(\mathbb{C})).$$

REMARK. Any scheme of finite type over $\mathbb{C}$ admits a stratification $(Z_i)_{i \in I}$ where each Z_i is smooth: Consider $X_{red} \subset X$ which is a closed subscheme. Then there is an open subset of X_{red} which is smooth. Let it be Z_0 and $Z_0 \hookrightarrow X$ is locally closed.

For $Hilb^n(Y)$ we stratify by partition: Let $\alpha \vdash n$ be a partition of n, $\alpha = (\alpha_1, \cdots, \alpha_n)$ where $\alpha_i \geq \alpha_{i+1}$ and $\alpha_r > 0$. Let $[Z] \in Hilb^n(Y)(k)$ where k is any field. Then we know that $Z = \mathcal{S}pec(B)$ for a finite k-algebra B.

PROPOSITION 10. *Every finite dimensional k-algebra B is (up to order) uniquely isomorphic to a product $B \cong B_1 \times \cdots \times B_r$ where B_i is a local finite dimensional k-algebra.*

PROOF SKETCH. **TO DO** □

So every k-valued point $\mathcal{S}pec\, k \to Hilb^n Y$ has an associated partition given by $(\dim_k(B_1), \cdots, \dim_k(B_n))$. Let $Hilb^n_\alpha Y$ be the set of points with their corresponding partition being α. We put a partial order on the set of all paritions of n by defining $\alpha \geq \beta$ if and only if α is a refinment of β. The question is whether $(Hilb^n_\alpha Y)_{\alpha \vdash n}$ is a stratification of $Hilb^n_\alpha Y$.

It suffices to prove the assertion locally; i.e to show that for any ring A, and any closed subscheme $Z \hookrightarrow Y_A$ the diagram

$$\begin{array}{ccc} Z & \longrightarrow & Y_A \\ & \searrow & \downarrow \\ & & \mathcal{S}pec\, A \end{array}$$

gives a stratification of $\mathcal{S}pec\, A$.

Recall that we had defined $Hilb^n_\alpha(\mathbb{A}^s)$ is another particular way too. If $[B_1, \cdots, B_s, v] \in Hilb^n_\alpha(\mathbb{A}^s)(k)$ then V decomposes to generalized simultaneous eigenspaces $\oplus_{i=1}^r V_i$ and $\alpha = (\dim V_1, \cdots, \dim V_r)$.

LEMMA 12. *If $I \subset k[z_1, \cdots, z_r]$ is the deal corresponding to $[B_1, \cdots, B_s, v_0]$ (meaning $I = \{f : f(B) = 0\}$) then $k[z]/I \cong \oplus_{i=1}^r V_i$ as k-algebras, and each V_i is local.*

This lemma shows that the two partitions are in fact the same!

LEMMA 13. *If $Z \to \mathcal{S}pec\, A$ is a finite, flat, rank n morphism, there is an affine covering $\mathcal{S}pec\, A_{f_i}$ of $\mathcal{S}pec\, A$ where $\langle f_1, \cdots, f_r \rangle = 1$ such that over $\mathcal{S}pec\, A_{f_i}$ if Z_i fits into the cartesian diagram*

$$\begin{array}{ccc} Z_i & \longrightarrow & \mathcal{S}pec\, A_{f_i} \\ \downarrow & & \downarrow \\ Z & \longrightarrow & \mathcal{S}pec\, A \end{array}$$

then there exists an embedding

$$\begin{array}{ccc} Z_i & \hookrightarrow & \mathbb{A}^n_{A_{f_i}} \\ & \searrow & \downarrow \\ & & \mathcal{S}pec\, A_{f_i} \end{array} .$$

PROOF. **TO DO** □

This last lemma also implies that the verification of out claimed stratification reduces to the proof that it is so for $Y = \mathbb{A}^n$. We proceed with the proof of the latter.

TO DO

We restate this result in the following

THEOREM 14.2. *$(Hilb^n_\alpha(Y))_{\alpha \vdash n}$ is a stratification of $Hilb^n Y$. Moreover each $Hilb^n_\alpha Y$ has a natural scheme structure induced by the flattening stratification.*

COROLLARY 5. $\chi(Hilb^n(Y)(\mathbb{C})) = \sum_{\alpha \vdash n} \chi(Hilb^n_\alpha(Y)(\mathbb{C}))$.

Now let $\alpha \vdash n$ be any partition. We don't in general have a morphism

$$\prod_{i=1}^{r} Hilb^{\alpha_i} Y \to Hilb^n Y$$

vai $(Z_1, \cdots, Z_r) \mapsto Z_1 \cup \cdots \cup Z_r$. The problem is that it is not in general the case that $Z_1 \cup \cdots \cup Z_r$ is a subscheme of Y. This however is the case when Z_i's are disjoint! So we pass to

$$\begin{array}{ccc} V & \hookrightarrow & \prod_{i=1}^{r} Hilb^{\alpha_i} Y \\ & \searrow & \downarrow \\ & & Hilb^n Y \end{array}$$

Let $V \subset \prod_{i=1}^{r} Hilb^{\alpha_i} Y$ be the set of all $Z_1, \cdots, Z_r$ such that $i \neq j$ implies $Z_i \cap Z_j = \varnothing$.

LEMMA 14. *V is an open subscheme.*

PROOF. **TO DO** □

So for $\mathcal{Spec}\,A \to V$ given by $(Z_1, \cdots, Z_r)$ the scheme $Z_1 \coprod \cdots \coprod Z_r$ defines a map $\mathcal{Spec}\,A \to Hilb^n Y$. Let Z_α denote the fiber product in the following cartesian diagram

$$\begin{array}{ccccc} Z_\alpha & \longrightarrow & V & \hookrightarrow & \prod Hilb^{\alpha_i} Y \\ \downarrow & & \downarrow & \swarrow & \\ Hilb^n_\alpha Y & \longrightarrow & Hilb^n Y. & & \end{array}$$

Here $Z_\alpha \to Hilb^n_\alpha Y$ is a Galois cover with Galois group $\mathrm{Aut}(\alpha)$. Then

$$\chi(Hilb^n Y) = \sum_{\alpha \vdash n} \chi(Hilb^n_\alpha Y) = \sum_{\alpha \vdash n} \frac{1}{\#\mathrm{Aut}(\alpha)} \chi(Z_\alpha).$$

To compute $\chi(Z_\alpha)$ we observe that it fits into another cartesian diagram:

$$\begin{array}{ccc} Z_\alpha & \longrightarrow & \prod_i Hilb^{\alpha_i}_{(\alpha_i)} Y \\ \downarrow & & \downarrow \\ Y^r_0 & \longrightarrow & Y^r \end{array}$$

$Hilb^m_{(m)} Y \to Y$ is a Zariskli locally trivial fibration with fibers F_n being the punctual Hilbert scheme which is a subscheme of $Hilb^n(\mathbb{A}^s)$ consisting of all subschemes of $\mathbb{A}^s$ supported at the origin, this corresponds to $\{(B_1, \cdots, B_s, v_0)\}$ such that all B_i's are nilpotent, $s = \dim Y$. Hence $\chi(Z_\alpha) = \chi(Y^r_0) \prod \chi(F_{\alpha_i})$.

15. Euler characteristic of schemes over $\mathbb{C}$

THEOREM 15.1. *If X is of finite type over $\mathbb{C}$ then $\chi(X(\mathbb{C})) = \chi_c(X(\mathbb{C}))$.*

REMARK. For any locally compact space X, $z \subset X$ closd, and $U = X - Z$, then $\chi_c(X) = \chi_c(U) + \chi_c(Z)$. This is not true for χ without the compact support restriction. For example $X = \mathbb{R}$, and $Z = \{0\}$, then $\chi(\mathbb{R}) = 0$, $\chi(Z) = 1$, $\chi(\mathbb{R} - \{0\}) = 2$, while $\chi_c(\mathbb{R}) = -1$, $\chi_c(Z) = 1$, $\chi_c(\mathbb{R} - \{0\}) = -2$.

PROOF. **TO DO** □

PROPOSITION 11. *If $\mathbb{G}_m$ over $\mathbb{C}$ act on the finite type $\mathbb{C}$-scheme X without fixed points then $\chi(X(\mathbb{C})) = 0$.*

PROOF. **TO DO** □

REMARK. The finite stabilizers were used in the proof to pretend $Z = X/\mathbb{G}_m$ is a "space" so that $\chi(Z)$ is fintie. The proof works for any connected algebraic group G if all stabilizers are finite. This will imply that there are no fixed points and hence the quotient stack $[X/G]$ is a Deligne-Mumford stack. Then $[X/G](\mathbb{C})$ is an orbifold and so $\chi([X/G](\mathbb{C}))$ is finite. We may then conclude that $\chi(X(\mathbb{C})) = 0$ due to the fact that $\chi(G) = 0$.

REMARK. Given a complex manifold $U \hookrightarrow Y$ being an analytic open immersion and $U \hookrightarrow \mathbb{A}^s$ again analytically, we get and induced diagram

$$\begin{array}{ccc} Hilb^n(U) & \xrightarrow{\text{open}} & Hilb^n Y \\ & \searrow^{\text{open}} & \\ & & Hilb^n(\mathbb{A}^s) \end{array}$$

We have to rely on the exitence of the analytic category which $X(\mathbb{C})$ in an object of for all finite-type $\mathbb{C}$-schemes X. We use this to prove that for an open subscheme $U \subset Y$, $Hilb^n(U)$ is an analytic subspace of $Hilb^n(Y)$ and is hence an "analytic space".

TO DO

The above argumetn shows that if $U \subset Y$ is open, there is an analytic open of $Hilb^n(Y)$ which represents $Hilb^n(U)$. And thus:

COROLLARY 6. *All analytic Hilbert schemes we need are representable.*

Let's calculate $\chi(Hilb^n(\mathbb{A}^s))$ now. The tool is the natural $T = (\mathbb{G}_m)^s$ action on $\mathbb{A}^s$ via weight $(1, \cdots, 1)$:

$$(\lambda_1, \cdots, \lambda_s)(a_1, \cdots, a_s) \mapsto (\lambda_1 a_1, \cdots, \lambda_s a_s).$$

The T also acts on $Hilb^n(\mathbb{A}^s)$

$$T(A) \times Hilb^n \mathbb{A}^s(A) \to Hilb^n(\mathbb{A}^s)(A) = \{(V, B_1, \cdots, B_s, v_0) : \text{ with conditions! } \}/\cong$$

$$(\lambda_1, \cdots, \lambda_s)(V, B_1, \cdots, B_s, v_0) \mapsto (V, \lambda_1 B_1, \cdots, \lambda_s B_s, v_0).$$

Recall that an action of $\mathbb{G}_m$ on $\mathcal{S}pec\, A$ if an only if A is graded by $\mathbb{Z}$. So the exitence of an action of $(\mathbb{G}_m)^s$ on $\mathbb{A}^s$ is equivalent to the existence of the s-fold grading on $\mathbb{C}[x_0, \cdots, x_s]$ where the i-th degree of a polynomial is the degree in $x_i, (1 \le i \le s)$. An element is homogeneous if and only if each variable occurs with the same power in each monomial, iff it is a monomial. Each graded piece is a 1-dimensional module and had a canonical basis (the monic).

The fixed points of $T(\mathbb{C}) = (\mathbb{C}^*)^s$ on $Hilb^n(\mathbb{A}^s)(\mathbb{C}) = \{I \le \mathbb{C}[x_1, \cdots, x_s] : \text{corank} I = n\}$ are given by :

$$\begin{aligned} I \text{ is fixed} &\Leftrightarrow I \text{ is homogeneous} \\ &\Leftrightarrow I \text{ is generated by homogeneous elements} \\ &\Leftrightarrow I \text{ is generated by monomials} \end{aligned}$$

in the latter case we say I is a monomial ideal. These are in one-to-one correspondence with the $s - D$ partitions of n (when $s = 2$ these would be the usual partitions of n, when $s = 3$ they are the 3D partitions of it, etc.). Thus the number of $T(\mathbb{C})$-fixed points of $Hilb^n(\mathbb{A}^s)(\mathbb{C})$ is equal to $P_s^{(n)}$, denoting the number of s-partitions of n. We conclude that

COROLLARY 7. $\chi(Hilb^n(\mathbb{A}^s(\mathbb{C}))) = \chi(\text{fixedpointset}) + \chi(\text{complement}) = P_n^{(s)}$.

There is also a generating function that these numbers fit in:

$$F^{(s)}(g) = \sum_{n=0}^{\infty} \chi(Hilb^n(\mathbb{A}^s(\mathbb{C})))t^n = \sum_{n=0}^{\infty} P_n^{(s)} t^n = \begin{cases} \frac{1}{1-t} & s = 1 \\ \prod_{k=1}^{\infty} \frac{1}{1-t^k} & s = 2 \\ \prod_{k=1}^{\infty} \frac{1}{(1-t^k)^k} & s = 3 \end{cases}$$

16. $Hilb^n(\mathbb{A}^3)$ as a critical locus

PROPOSITION 12. *M is a smooth scheme.*

PROOF. We know that $M = \widetilde{M}/\mathrm{Gl}_n$ where

$$\widetilde{M} = \{(B_1, B_2, B_3, v_0) : \text{ stability condition } \} \subset M_{n\times n}^3 \times \mathbb{A}^n$$

is an open subscheme in some basis chosen and has dimension $3n^2 + n$, hence in particular smooth.

$$\begin{array}{ccc} \mathrm{Gl}_n \times \widetilde{M} & \xrightarrow{act} & \widetilde{M} \\ {\scriptstyle pr}\downarrow & & \downarrow{\scriptstyle \pi} \\ \widetilde{M} & \xrightarrow{\pi} & M \end{array}$$

We know that this is cartesian, that π is sheaf surjective, and M is a sheaf. From these facts it follows that M is a smooth scheme □

And $\widetilde{M} \to M$ is a principal Gl_n-bundle. $Hilb^n(\mathbb{A}^3) \hookrightarrow M$ is a closed subscheme given by $[B_i, B_j] = 0$ and is in general very singular.

The goal is now to show that there exists a regular function $f \in \mathcal{O}(M)$ (i.e. a map $M \xrightarrow{f} \mathbb{A}^1$) such that $Hilb^n(\mathbb{A}^3) = Z(df)$. Moreover $Hilb^n(\mathbb{A}^3) \subseteq Z(f) = f^{-1}(0)$. So $Hilb^n(\mathbb{A}^3)$ is the singular locus of the hypersurface $Z(f) \subset M$.

$$f((V, B_1, B_2, B_3, v_0)) := \mathrm{tr}(B_1[B_2, B_3]).$$

where V is a finite flat module over a $\mathbb{C}$-algebra A.

16.1. Computing df. Work on $\widetilde{M} \subseteq M^3_{n\times n} \times \mathbb{A}^n$. Consider a $\mathbb{C}[\varepsilon]$-valued point of $\widetilde{M}$:

$$b + \varepsilon x = (B_1 + \varepsilon X_1, B_2 + \varepsilon X_2, B_3 + \varepsilon X_3, v_0 + \varepsilon x_0)$$

Then we have,

$$\begin{aligned} f(b+\varepsilon x) &= \operatorname{tr}((B_1 + \varepsilon X_1)[B_2 + \varepsilon X_2, B_3 + \varepsilon X_3]) \\ &= \operatorname{tr}(B_1[B_2, B_3]) + \varepsilon \{\operatorname{tr}(X_1[B_1, B_3]) + \operatorname{tr}(B_1[X_2, B_3]) + \operatorname{tr}(B_1[B_2, X_3])\} \end{aligned}$$

Thus $df_b : M^3_{n\times n} \times \mathbb{C}^n \to \mathbb{C}$ is the $\mathbb{C}$-linear mapping

$$(x_1, x_2, x_3, v_0) \mapsto \operatorname{tr}(X_1[B_1, B_3]) + \operatorname{tr}(B_1[X_2, B_3]) + \operatorname{tr}(B_1[B_2, X_3]).$$

The partial derivatives with respect to the standard coordinates are obtained by setting

$$(X_1, X_2, X_3, x_0) = \begin{cases} (E_{ij}, 0, 0, 0) \\ (0, E_{ij}, 0, 0) \\ (0, 0, E_{ij}, 0) \\ (0, 0, 0, e_i) \end{cases}$$

We know that

$$\begin{aligned} \operatorname{tr}(E_{ij}[B_1, B_3]) &= \ (j,i) \text{ entry of } [B_1, B_3] \\ \operatorname{tr}(B_1[E_{ij}, B_3]) &= -\ (j,i) \text{ entry of } [B_1, B_3] \\ \operatorname{tr}(B_1,[B_2, E_{ij}]) &= \ (j,i) \text{ entry of } [B_1, B_2] \end{aligned}$$

So the Jacobian is given by

$$df = ([B_1, B_3]^t, [B_3, B_1]^t, [B_1, B_2]^t, 0).$$

On $\widetilde{M}$ have $\widetilde{f} : \widetilde{M} \to \mathbb{A}^1$ with $Z(df) = \widetilde{H}$. Then follows that for $f : M \to \mathbb{A}^1$ we have

$$Z(df) = Hilb^n(\mathbb{A}^3) = M.$$

Remark. $Z(df) \subset Z(f)$ by choice of f. So $Hilb^n(\mathbb{A}^3)$ is the singular locus of the hyperplane $f^{-1}(0) \subset M$. This is NOT true in higher dimensions!

17. Singular loci of hypersurfaces

Let $f : \mathbb{C}^m \to \mathbb{C}$ be a holomorphic function. $V = f^{-1}(0)$ and let $H = Z(df)$. We assume $f \in \langle \frac{\partial f}{\partial x^i} \rangle$. Thus H will be the singular locus of V. Assume p is the origin of $\mathbb{C}^m$ and let

$$\nu_H(p) = (-1)^m(1 - \chi(F_p)).$$

If p is an isolated singularity of V (isolated point of H), then F_p is homotopy equivalent to a bonquet of $(m-1)$-spheres, and

$$\nu_H(p) = (-1)^m(1 - \chi(\text{ bonquet})) = (-1)^m(1 - (1 - (-1)^{m-1}.\#\text{ spheres })) = \#\text{spheres}$$

and this is known as the Milnor number of the singularity.

REMARK. The Milnor number is equal to $\dim_{\mathbb{C}} \mathcal{O}(H)$ so $\nu_H(p)$ is a natural generalization of Milnor number to non-isolated singularities.

We now define the weighted Euler characteristic χ^B of H with weight ν_H to be

$$\chi(H, \nu_H) = \sum_{H_\alpha} \nu_H|_{H_\alpha} \chi(H_\alpha)$$

where $\{H_\alpha\}$ is an stratification of H and ν is constant on each stratum. This is also the Donaldson-Thomas invariant of H.

For $p \in M^n$ choose an analytic coordinate chart around $p \in U \subset M^n$ and $U \hookrightarrow \mathbb{C}^{2n^2+n}$.

THEOREM 17.1. *Milnor fiber is invariant under biholmorphic maps:*

$$\begin{array}{ccc} p \in \mathbb{C}^n & \xrightarrow{\text{biholo.}} & \mathbb{C}^n \ni q \\ & g \searrow \quad \swarrow f & \\ & \mathbb{C} & \end{array}$$

Then $F_p^{(g)} \cong F_q^{(f)}$ *is a homeomorphism.*

So by this fact every point $p \in M$ has a well-defined Milnor fiber F_p, hence we get $\nu : M \to \mathbb{Z}$ via $p \mapsto (-1)^{2n^2+n}(1-\chi(F_p)) = (-1)^n(1-\chi(F_p))$ as a $\mathbb{Z}$-valued function on M which vanished outside of $Hilb^n(\mathbb{A}^3)$.

18. Behrend function for general Hilbert schemes of n points

The goal is to define and compute $\nu : Hilb^n Y \to \mathbb{Z}$ for any smooth $\mathbb{C}$-scheme Y of dimension 3. Let $p = [Z] \in Hilb^n Y(\mathbb{C})$ for some subscheme $Z \hookrightarrow Y$. Choose holomorphic coordinate charts for Y, $\{U_i\}$. If $Z = Z_1 \coprod \cdots \coprod Z_r$ where length of Z_i is α_i and $\alpha \vdash n$ is a partition of n. Assume $Z_j \subset U_j$ for all j and for simplicity suppose that $U_1, \cdots, U_r$ are pairwise disjoint.

$$\begin{array}{ccccc} p = ([Z_1], \cdots, [Z_r]) \in \prod_{i=1}^r Hilb^{\alpha_i} U_i & \xrightarrow[\text{open}]{} & Hilb^n Y \ni [Z] & & \\ \downarrow & & & & \\ \prod_{i=1}^r Hilb^{\alpha_i} \mathbb{A}^3 & \xrightarrow{proj_i} & Hilb^{\alpha_i} \mathbb{A}^3 & \xrightarrow{\nu} & \mathbb{A}^1 \end{array}$$

This shows that $\nu_{Hilb^n Y}(p) = \prod_{i=1}^r \nu_{Hilb^{\alpha_i} \mathbb{A}^3}([Z_i])$.

Is this well-defined? For this we need independence of choice of charts and independence of partition $Z_1 \coprod \cdots \coprod Z_r$.

TO DO

REMARK. ν_{Hilb^nY} is constructible, i.e. there is a stratification of $Hilb^nY$ such that ν is constant on each stratum.

This follows from the corresponding fact for Milnor fibers! Now $\chi(Hilb^nY, \nu)$ can be defined analogous to the above and is well-defined.

THEOREM 18.1. $\chi(Hilb^nY, \nu) = (-1)^n\chi(Hilb^nY)$.

So for any smooth scheme Y of dimension 3 over $\mathbb{C}$:

$$\sum_{n=0}^{\infty}\chi(Hilb^nY, \nu)t^n = \sum_{n=0}^{\infty}\chi(Hilb^nY)(-t)^n = \left(\prod_{k=1}^{\infty}\frac{1}{(1-(-t)^k)^k}\right)^{\chi(Y)}.$$

$$\begin{aligned}\chi(Hilb^nY, \nu) &= \sum_{\alpha\vdash n}\chi(Hilb^n_\alpha Y, \nu_{Hilb^nY}) \\ &= \sum_{\alpha\vdash n}\frac{1}{\#\mathrm{Aut}(\alpha)}\chi(Z_\alpha, \nu_{Hilb^nY}) \\ &= \sum_{\alpha\vdash n}\frac{1}{\#\mathrm{Aut}(\alpha)}\chi(Z_\alpha, \prod\nu_{Hilb^{\alpha_i}\mathbb{A}^3}) \\ &= \sum_{\alpha\vdash n}\frac{1}{\#\mathrm{Aut}(\alpha)}\chi(Y_0^r)\prod_{i=1}^{r}\chi(F_{\alpha_i}, \nu_{Hilb^{\alpha_i}\mathbb{A}^3})\end{aligned}$$

So until now we have reduced this computation to $Hilb^n(\mathbb{A}^3)$.

Recall that $\mathbb{G}_m^3$ acts on M^n. Let $T\subset\mathbb{G}_m^3$ be defined by $\lambda_1\lambda_2\lambda_3 = 1$ and this is isomorphic to $\mathbb{G}_m^2$.

f is constant on T-orbits. As a result T acts on $Hilb^n(\mathbb{A}^3)$ and hence as before

$$\chi(Hilb^n\mathbb{A}^3, \nu) = \sum_p \nu(p)$$

where p ranges on the fixed points of the action of T.

Suppose $p\in Hilb^n(\mathbb{A}^3)$ is a fixed point of the action of T. Since the T-action on $\mathbb{A}^3$ is compatible with the action of it on points of $Hilb^n(\mathbb{A}^3)$, the fact that p is fixed implies that the support of the corresponding closed subscheme of $\mathbb{A}^3$ is also fixed. But the action of T on $\mathbb{A}^3$ has only the origin as fixed point. Thus the closed subscheme corresponding to p is supported only at the origin. Suppose that the corresponding ideal is $I\subset\mathbb{C}[x,y,z]$. Then by general theory on graded rings, I has to be generated by eigenvectors of the action of T. But these are all of the form $g(xyz)m(x,y,z)$ where m is a monomial. We can assume that $g(0)\neq 0$. Thus $g\notin I$. Hence $V(\langle g\rangle + I) = \varnothing$. Hence $\alpha g + \beta = 1$, and $m\alpha g + m\beta = m$. So the left hand side is also in I, i.e. $m\in I$. Thus I is generated by its monomials. Hence is any monomial ideal.

Now from the following more general example we have that $\chi(F_p) = 0$. Thus $\nu(p) = (-1)^{\dim M^n}(1-0) = (-1)^n$,

$$\chi(Hilb^n\mathbb{A}^3, \nu) = \sum_{p \text{ fixed}} \nu(p) = (-1)^n P_n^{(3)}.$$