Algebraic K-theory
Math 600D - Fall 2011

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Lecture notes by Pooya Ronagh

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CHAPTER 1

Zeroeth K-groups

1. K-theory of an abelian category

Recall that if $R$ is a ring, an $R$-module $P$ is said to be projective if any of the following equivalent conditions are satisfied:

(1) $P$ is a direct summand of free modules,

(2) $0 \to M \to N \to P \to 0$ splits for any sequence of $R$-modules,

(3) the functor $\text{Hom}_R(P, -)$ is exact,

(4) the following diagram always completes

$$
\begin{array}{ccc}
X & \to & Y \\
\downarrow & & \downarrow \\
\rotatebox{90}{$\to$} & & \rotatebox{90}{$\to$} \\
P & & \\
\end{array}
$$

The projective dimension, $\text{pd}_R(M)$, of the $R$-module $M$ is the minimum length of all projective resolutions. In particular $\text{pd}_R M = 0$ whenever $M$ is projective. The global dimension (or homological dimension) of $R$ is the supremum over $\text{pd}_R(M)$ of all $R$-modules $M$. A good example to recall is that if $R$ is a regular local ring, then $\text{gl.dim}_R < \infty$ and is equal to its Krull dimension. We will use the notations $R - \text{Mod}$, $R - \text{mod}$ and $\mathcal{P}_R$ respectively for the categories of all, finitely generated, and finitely-generated projective $R$-modules.

By an abelian category we mean a small preadditive category (i.e. $\text{Hom}(A, B)$ has the structure of an abelian group, compatible with category structure), that has finite (co-)products, has a zero object (i.e. both initial and terminal), is (co-)normal (i.e. every mono(epi-)morphism is the (co-)kernel of some morphism), and finally has epi-mono factorization:

$$
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow & \swarrow & \downarrow \\
\text{im}(f) & & \\
\end{array}
$$
In an abelian category, \( \mathcal{A} \), the abelian group \( K_0(\mathcal{A}) \) is one generated by isomorphism classes of objects of the category with the relations

\[
[B] = [A] + [C], \quad \text{for any short exact sequence } 0 \to A \to B \to C \to 0.
\]

This group is universal in the sense that for any abelian group \( G \), and group homomorphism \( f : \text{ob}(\mathcal{A}) \to G \) respecting the relations \( f(B) = f(A) + f(C) \) for any short exact sequence as above, then there is a unique group morphism completing

\[
\begin{CD}
\text{ob}(\mathcal{A}) @>>> K_0(\mathcal{A}) \\
@VfVV @. \\
G @. \\
\end{CD}
\]

**Lemma 1.** If \( \mathcal{A} \) has countable direct sums then \( K_0(\mathcal{A}) = 0 \).

**Proof.** Let \( B = \bigsqcup_{\infty} A \) for any object \( A \). Then \( A \oplus B = B \) implying \( [A] = 0 \). \( \square \)

If \( \mathcal{A} \) and \( \mathcal{B} \) are two abelian categories and \( T : \mathcal{A} \to \mathcal{B} \) is an exact functor we get an induced group homomorphism \( K_0(\mathcal{A}) \to K_0(\mathcal{B}) \). One therefore concludes that \( K_0 \) is a co-variant functor from the category of abelian categories and exact functors between them to the category of abelian groups.

**Corollary 1.** If \( \mathcal{A} \) and \( \mathcal{B} \) are equivalent abelian categories, then \( K_0(\mathcal{A}) = K_0(\mathcal{B}) \).

It is immediate that \( K_0(ab) \), that of the category of abelian groups is isomorphic to \( \mathbb{Z} \) and the isomorphism is given by the rank of the abelian group. If \( \mathcal{A} \) is however the category of finite abelian groups then \( K_0(A) \) is the free abelian group with basis \( [\mathbb{Z}/p] \)'s. A more general observation is the following

**Lemma 2.** If \( \mathcal{A} \) is an abelian category with Jordan-Holder filtrations, i.e.

\[
A = A_n \supseteq A_{n-1} \supseteq \cdots \supseteq A_1 \supseteq A_0 = 0
\]

with simple factors, \( A_i/A_{i-1} \), then

\[
K_0(\mathcal{A}) = \bigoplus_{\text{simple } S} [S].
\]

**Lemma 3.** Two objects \( A, B \) in the abelian category \( \mathcal{A} \) have the same class in \( K_0(\mathcal{A}) \) if and only if there is \( C \) in \( \mathcal{A} \) such that \( A \oplus C = B \oplus C \).

In particular, this motivates the jargon *stably isomorphic* for \( R \)-modules: \( M \) and \( N \) are stably isomorphic \( R \)-modules if

\[
M \oplus R^k \not\cong N \oplus R^k
\]

for some integer \( k > 0 \). We say \( M \) is *stably free* if

\[
M \oplus R^k \not\cong R^n
\]

for some integers \( k, n > 0 \).
2. $K_0$ of local rings

In what follows we do not need $R$ to be necessarily a commutative ring. Our goal is to study the Grothendieck group,

$$K_0(R) := K_0(\mathcal{P}_R),$$

of the category of projective $R$-modules. It is for instance immediate that if $R$ is any PID, then rank induces $K_0(R) \cong \mathbb{Z}$.

Jacobson radical is the intersection of all maximal ideals as in the commutative case. Then the following version of Nakayama lemma is handy:

**Theorem 2.1 (Nakayama lemma).** Let $I$ be an ideal of the ring $R$ containing the Jacobson radical. Suppose $M$ is a finitely-generated $R$-module. Then $M/IM = 0$ implies $M = 0$.

**Theorem 2.2 (Kaplansky).** Let $R$ be a local ring and $P$ a projective $R$-module. Then $P$ is free.

**Proof.** Let $J = J(R)$ be the Jacobson radical of $R$. Since $R$ is assumed to be a local ring, this is the unique maximal ideal of $R$. Hence $D = R/J(R)$ is a division ring. Let $M$ be a finitely generated projective $R$-module. In particular say

$$M \oplus N \cong R^n.$$

Then $M/IM$ and $N/JN$ are $D$-vector spaces. We now fix $e_1, \ldots, e_m$ and $e'_1, \ldots, e'_s$ to form bases of the former vector spaces. By Nakayama’s lemma these lift to generators $e_1, \ldots, e_m$ and $e'_1, \ldots, e'_s$ of $M$ and $N$. It now suffices to show that

$$\{x_1, \ldots, x_n\} = \{e_1, \ldots, e_m, e'_1, \ldots, e'_s\}$$

form a basis of $R^n$. Let $f_1, \ldots, f_{m+s}$ be the standard basis of $M \oplus N \cong R^n$. So we already have

$$f_i = \sum a_{ij}x_i, x_i = \sum b_{ij}f_j$$

for all $i = 1, \ldots, n$. By the notation $A = (a_{ij})$ and $B = (b_{ij})$ we have $AB = I_n$. Recall now that

$$J(R) = \{x \in R : 1 - ax \text{ is invertible for any } a \in R\}.$$ 

It follows now that $BA - I \in M_n(J(R))$ proving our claim. \qed
Another interesting variant of $K_0$ is

$$G_0(R) := K_0(R - \text{mod}).$$

**Remark.** If $f : R \to R'$ is a ring homomorphism we get an induced group homomorphism

$$f_* : K_0(R) \to K_0(R')$$

if moreover $R'$ is $R$-flat then we have an induced homomorphism

$$f_* : G_0(R) \to G_0(R')$$

as well. If $R'$ is a finitely-generated $R$-module, then the forgetful functor $R' - \text{mod} \to R - \text{mod}$ induced

$$\text{res} : G_0(R') \to G_0(R)$$

and if $R'$ is moreover $R$-projective, we get an induced

$$\text{res} : K_0(R') \to K_0(R).$$

Our final remark is that the natural inclusion $P \subset R - \text{mod}$ induces the so called Cartan homomorphism

$$c_0 : K_0(R) \to G_0(R).$$

**Theorem 3.1 (Idempotent lifting).** Let $R$ be a ring and $J$ a two-sided ideal of $R$ contained in the Jacobson radical. Let $\overline{R} = R/J$. Then the quotient map induces a functor

$$\mathcal{P}_R \to \mathcal{P}_{\overline{R}}$$

which is full, additive and satisfies the following: if $f : P \to Q$ is a morphism in $\mathcal{P}_R$ such that $\overline{f} : \overline{P} \to \overline{Q}$ is an isomorphism then $f$ is also an isomorphism.

This is easy to prove. In particular we have

**Theorem 3.2.** If $R$ is $J$-adicly complete (i.e. $\varinjlim R/J^n \cong R$), then the above functor $\mathcal{P}_R \to \mathcal{P}_{\overline{R}}$ is bijective.

**Proof.** Given $Q \in \mathcal{P}_{\overline{R}}$ we look for a lift of it $P \in \mathcal{P}_R$. Since $Q$ is projective we may think of it as the image of an idempotent:

$$Q = \text{im}(\overline{p}), \overline{p} \in M_n(\overline{R}) = \text{End}_{\overline{R}} \overline{R}^n, \overline{p}^2 = \overline{p}.$$

All we have to do is to lift $\overline{p}$ to idempotent $p \in M_n(R)$. Let $A = M_n(R), \overline{A} = M_n(R/J) = M_n(R)/M_n(J)$. Given any $a \in A, n > 0$ we have

$$1 = (a + (1 - a))^2 = \sum_{0}^{2n} \binom{2n}{j} a^{2n-j}(1-a)^j.$$

But

$$f_n(a) = \sum_{0}^{n} \binom{2n}{j} a^{2n-j}(1-a)^j \equiv a \mod a^n A.$$
4. K-theory of Dedekind domains

Recall that a Dedekind domain $R$ is an intergrally closed noetherian domain of Krull dimension 1.

**Example 4.1.** Examples are $\mathbb{Z}$, $k[x]$, ring of integers in an algebraic number field. Let $K$ be a finite extension of $\mathbb{Q}$ and

$$
\mathcal{O}_K = \{ \theta \in K : f(\theta) = 0 \text{ for some monic polynomial } f(x) \in \mathbb{Z}[x] \}
$$

be the integral closure of $\mathbb{Z}$ in $K$

$$
\begin{array}{ccc}
\mathcal{O}_K & \longrightarrow & K \\
\downarrow & & \downarrow \\
\mathbb{Z} & \subseteq & \mathbb{Q}
\end{array}
$$

Then $\mathcal{O}_K$ is a Dedekind domain. If $R$ is a noetherian, integrally closed ring and $\mathfrak{p}$ is a prime ideal of height one, then $R_{\mathfrak{p}}$ is also a Dedekind domain.

So let $R$ be a Dedekind domain and $K$ its function field.

**Definition 1.** A fractional ideal of $K$ is a nonzero $R$-submodule $I$ of $F$ such that there is an element $a \in I$ with $aI \subseteq R$.

Fractional ideals form an abelian monoid under multiplication with (1) the identity element. If $I$ is a fractional ideal, there exists a fractional ideal

$$
I^{-1} = \{ a \in K : aI \subseteq R \}
$$

and that $II^{-1} = R$. So the fractional ideals of $R$ form a group and the principal fractional ideals form a subgroup.

**Definition 2.** The class group $C(R)$ for a Dedekind domain $R$ is defined to be

$$
C(R) = \frac{\text{the group of fractional ideals}}{\text{subgroup of principal fractional ideals}}
$$

**Theorem 4.1.** If $R$ is a Dedekind domain, then every fractional ideal is finitely generated and projective.
5. K-THEORY OF RINGS

Proof. Let I be a fractional ideal, since $I^{-1}I = R$ there are elements $x_1, \ldots, x_n \in I^{-1}$ and $y_1, \ldots, y_n \in I$ such that

$$\sum_{i=1}^{n} x_i y_i = 1$$

If $b \in I$, then $b = \sum (bx_i) y_i$ with $bx_i \in II^{-1} = R$. Hence $y_1, \ldots, y_n$ generate I.

Consider the homomorphism $R^n \to I$ via $e_i \mapsto y_i$. This has a splitting with right inverse

$$b \mapsto (bx_1, \ldots, bx_n)$$

Hence I is a direct summand of a free modli and therefore projective. □

Corollary 3. If R is a Dedekind domain, then every finitely generated projective R-modulie is isomorphic to a direct sum of ideals. In particular the isomorphism classes of ideals generate $K_0(R)$.

Proof. Let $M \in \mathcal{P}_R$. Then $M \subseteq R^n$. Proof is by induction on $n$. Let $\pi : R^n \to R$ be the projection to the last factor. Then $\pi(M) \subseteq R$, hence $\pi(M)$ is an ideal in $R$. If $\pi(M) = 0$ we are done otherwise, we may assume that

$$0 \neq \pi(M) = I \subseteq R.$$ 

As I is projective we get $M \cong \ker \pi|_M \oplus I$. But $\ker \pi \subseteq R^{n-1}$ and we are done by induction hypothesis. □

It is now easy to prove

Theorem 4.2. If R is a Dedekind domain, then $K_0(R) \cong \mathbb{Z} \oplus C(R)$.

Proof idea. Any finitely-generated projective module, P, of rank $n$ can be expressed as $R^{n-1} \oplus I$ for a fractional ideal I. The mapping $K_0(R) \to \mathbb{Z}$ is just the rank map. □

5. K-theory of rings

Recall that $K_0(R) = K_0(\mathcal{P}_R)$ and $G_0(R) = K_0(R - \text{mod})$.

Theorem 5.1 (Devisage, Hellor). Let $\mathcal{A}$ be an abelian category, $\mathcal{C}$ and $\mathcal{B}$ full subcategories with $\mathcal{C} \subseteq \mathcal{B}$ and such that

1. $\mathcal{C}$ is closed in $\mathcal{A}$ with respect to subobjects and quotient objects.

2. Every object of $\mathcal{B}$ has a finite filtration with all quotients in $\mathcal{C}$.

Then the canonical map $K_0(\mathcal{C}) \to K_0(\mathcal{B})$ is an isomorphism.
Proof. The inverse to the natural map $K_0(C) \rightarrow K_0(B)$ is defined by $f : K_0(B) \rightarrow K_0(C)$ as follows. Let $B \in B$ and

$$B = B_0 \supset B_1 \supset \cdots \supset B_n = 0$$

such that all factors are in $C$. Check that $[B] \mapsto \sum [B_i/B_{i+1}]$ works. \qed

Corollary 4. Let $\mathcal{A}$ be an abelian category in which each object has finite length (with respect to simple objects). Then $K_0(\mathcal{A})$ is the free abelian group on $[S]$ where $S$ varies over a set of representatives of the simple objects of $\mathcal{A}$.

Theorem 5.2. Let $\mathcal{A}$ be an abelian category and $\mathcal{P}$ be the full subcategory of projective objects (i.e. those for which $\text{Hom}(P, -)$ is exact). If $\text{pd} A$ (via resolution by projectives) is finite for every object $A \in \mathcal{A}$, then the natural map

$$I : K_0(\mathcal{P}) \rightarrow K_0(\mathcal{A})$$

is an isomorphism.

Proof. Check that $A \mapsto \sum (-1)^i P_i$ works. For this we need Schanuel's lemma that we recall here.

Lemma 4 (Schanuel's). Let $\mathcal{A}$ be an abelian category and

$$0 \rightarrow B \rightarrow P \rightarrow A \rightarrow 0 \text{ and } 0 \rightarrow B' \rightarrow P' \rightarrow A' \rightarrow 0$$

are exact sequences with $A \cong A'$ and $P, P'$ projective. Then $B \oplus P' \cong B' \oplus P$.

This proves independence of the projective resolution once we pass to long exact sequences from this. In fact let $P^* \rightarrow A$ and $P'^* \rightarrow A$ are two resolutions. By Schanuel's lemma $P_n \oplus P'_{n-1} \oplus \cdots$ is isomorphic to $P''_n \oplus P'_{n-1} \oplus \cdots$. Thus

$$\sum_{\text{odd}} P_i \oplus \sum_{\text{even}} P'_k \cong \sum_{\text{even}} P_i \oplus \sum_{\text{odd}} P'_k.$$

Next step is to show that this map is independent of the choice of representative for the class of $A$. \qed

Corollary 5. If $R$ is regular (i.e. every finitely generated $R$-module has a finite projective resolution) then $K_0(R) \rightarrow G_0(R)$ is an isomorphism.

Corollary 6. If $R$ is a regular commutative ring, then $K_0(R) \rightarrow G_0(R)$ is an isomorphism.

When working with commutative rings the notion of regularity is given equivalently by

Remark. A local ring $(A, \mathfrak{m})$ is regular whenever krull dim $A = \text{dim}_k \mathfrak{m}/\mathfrak{m}^2$. A ring is regular if $A_p$ is regular for all prime ideals in $A$ (equivalently if so for maximal ideals).
In general if $R$ is a ring (domain), we can define $\text{Pic}(R)$ to be the group generated by the set of all invertible ideals of $R$. Then we always have a surjection

$$K_0(R) \twoheadrightarrow \text{Pic}(R).$$

It is not always the case that $K_0(R) = G_0(R)$. For instance let $R = k[t^2, t^3] \subset k[t]$. Then $\text{Pic}(A) \cong k_+$, the additive group of $k$.

REMARK. The above remarks motivate a general definition of Krull dimension for an $R$-module. And this is given as

$$\text{dim}_R M := \text{dim}(R/\text{Ann}_R M).$$

REMARK. $G_0(A) = K_0(A - \text{mod})$ has two broad classes: that of the free modules (a $\mathbb{Z}$-contribution) and those with finite length (giving a free group on simple modules).

6. K-theory of rings of polynomials

The question we are going to consider is conditions under which $K_0(R[t]) \cong K_0(R)$. This is not always the case but, we will see that this holds when $R$ has finite homological dimension. We will need the following tools:

**Theorem 6.1 (Hilbert’s Syzygy).** If $R$ is a ring such that $\text{gl.dim} R \leq n$, then $R[t]$ has $\text{gl.dim} R[t] \leq n + 1$.

**Proposition 1.** If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of $R$-modules, then

$$\text{pd} M_2 \leq \max\{\text{pd} M_1, \text{pd} M_2\}$$

and equality holds if and only if $\text{pd} M_3 \neq \text{pd} M_1 + 1$.

**Proposition 2.** $\text{pd}_k M \geq n$ then $\text{Tor}_i^R(M, N) = 0$ for all $i \geq n + 1$ and for all modules $N$.

This proves

**Lemma 5.** Let $x$ be a (central) nonzero divisor in a ring $S$. If $M$ is a nonzero $S/x$-module with $\text{pd}_{S/x} M < \infty$ then $\text{pd}_X M = 1 + \text{pd}_{S/x} M$.

**Theorem 6.2.** If $R$ is a ring with $\text{gl.dim}_R \geq n$ then $R[t]$ has $\text{gl.dim} \leq n + 1$.

**Proof.** Step 1. Let $S = R[x]$. As $S$ is $R$-flat, hence $\text{pd}_S M[x] \leq \text{pd}_R M$. For projective resolution of $R$-modules

$$P^* \Rightarrow M$$

we tensor by $R[x]$ and get

$$R[x] \otimes_R P^* \Rightarrow R[x] \otimes_R M$$

which is a projective resolution of $R[x] \otimes_R M \cong M[x]$. Check that the previous lemma implies that

$$\text{gl.dim} R[x] \geq 1 + \text{gl.dim} R.$$
Step 2. Let $M$ be an $S$-module and consider $M_R$. Consider the $S$-modules
\[ 0 \to S \otimes_R M \xrightarrow{\beta} S \otimes_R M_R \xrightarrow{\mu} M \to 0 \]
where $\mu : s \otimes m \to sm$ and $\beta : s \otimes m \mapsto s(x \otimes m - 1 \otimes xm)$. (Check that this is exact).

Step 3. Use comparison statement for projective dimension on exact sequence to conclude that $\text{pd}_S M \leq 1 + \text{pd}_R M$. So
\[ \text{gl.dim} S \leq 1 + \text{gl.dim} R \]
and we are done. \qed

**Theorem 6.3.** For noetherian $R$ with finite global dimension, we have
\[ K_0(R[t]) \cong K_0(R) \cong K_0(R[t, t^{-1}]). \]

**Proof.** We have split short exact sequences
\[ 0 \to tR[t] \to R[t] \to R \to 0 \]
and
\[ 0 \to (t-1)R[t, t^{-1}] \to R[t, t^{-1}] \to R \to 0 \]
of $R$-modules. Hence $K_0(R[t])$ contains $K_0(R)$ as a direct summand.

As $S = R[t]$ is $R$-flat, we have $f_* : G_0(R) \to G_0(S)$. Also $\text{hd}_{R[t]} R = 1$. Now take a short exact sequence
\[ 0 \to M_1 \to M_2 \to M_3 \to 0 \]
of $R[t]$-modules and we derive with tensoring with $R$ as an $R[t]$-module. Because of projective dimension of $R$ higher Tors vanish and our long exact sequence reduces to
\[ \begin{align*}
0 & \to \text{Tor}_1^{R[t]}(R, M_1) \to \text{Tor}_1^{R[t]}(R, M_2) \to \text{Tor}_1^{R[t]}(R, M_3) \\
& \to R \otimes_{R[t]} M_1 \to R \otimes_{R[t]} M_2 \to R \otimes_{R[t]} M_3 \to 0
\end{align*} \]

We will show that $f_*$ has a right inverse, $\varphi$ and then prove that $f_*$ is also injective: $\varphi : G_0(R[t]) \to G_0(R)$ is define by
\[ [M] \mapsto [R \otimes_{R[t]} M] - [\text{Tor}_1^{R[t]}(R, M)] \]
and is well-defined by our long exact sequence. Now
\[ M \xrightarrow{f_*} [R[t] \otimes_R M] \xrightarrow{\varphi_*} [R \otimes_{R[t]} M] \to [\text{Tor}_1^{R[t]}(R, R[t] \otimes_R M)] \]
so $f_*$ is a split surjection. It remains to show it is injective.
For this part we use the following observation of Grothendieck: When $R$ is increasing-filtered $R = \text{inj lim } R_n$, we get a grading on $R$ by factors of the filtration.

\[ G_0(R) \otimes_{\mathbb{Z}} \mathbb{Z}[t] \cong G_0^{gr}(R) \]

and we have

\[ K_0(grR - \text{mod}) \]

\[ G_0^{gr}(R) \xrightarrow{\theta} G_0^{gr}(R[t,s]) \]

\[ \text{forget} \]

\[ \psi_* \]

\[ G_0(R) \xleftarrow{\psi_*} G_0(R[t]) \]

So the completion of the proof is by showing that this is a commutative diagram, $\psi_*$ is surjective and therefore that $\theta$ is an isomorphism. \qed

**Corollary 7.** If $\text{gl dim } R < \infty$, so is $\text{gl dim } R[x_1, \ldots, x_n]$.

**Corollary 8.** For a field $k$, $K_0(k[x_1, \ldots, x_n]) \cong G_0(k[x_1, \ldots, x_n]) \cong \mathbb{Z}$.

So this shows that every projective $k[x_1, \ldots, x_n]$-module $P$ is stably free, i.e.

\[ P \oplus R^k \cong R^{k+t}. \]

Can we do a cancellation then? The ideas to come are

- Quillen/Suslin: Stably free $k[x_1, \ldots, x_n]$-projective moduels are free.

- Bass cancellation: Let $R$ be a commutative noetherian domain of Krull dimension $d$. The every stably free $R$-module of rank $> d$ is a free module over $R$.

### 7. K-theory of topological spaces

Let $X$ be a compact, Hausdorff topological space and $\text{Vect}_F(X)$ be the commutative monoid of isomorphism classes of $F$-vector bundles over $X$, under the Whitney sum operation and $X$ being the unit. Here $F$ is the field $\mathbb{R}$ or $\mathbb{C}$. The topological K-theory of $X$ is defined by

\[ K^0_F(X) = K_0(\text{Vect}_F(X)) \]

Then $X \mapsto \text{Vect}_F(X)$ is a contravariant functor from the category of compact Hausdorff spaces to the category of abelian groups.

**Theorem 7.1 (Swan).** Let $R = C^F(X)$ be the ring of continuous $F$-valued functions on $X$ nad $\Gamma(X,E)$ be the $R$-module of sections of $E$, then

1. $\Gamma(X,E)$ is finitely generated and projective over $R$.

2. Conversely, every finitely-generated projective $R$-module arises this way.
(3) The map \( E \mapsto \Gamma(X, E) \) induces an isomorphism of categories between \( \text{Vect}_F(X) \) and \( \mathcal{T}_R \), hence an isomorphism
\[
K^0_F(X) \rightarrow K_0(R).
\]

8. K-theory of schemes

Let \( \text{Vect}(X) \) be the category of algebraic vector bundles on the ringed space \( (X, \mathcal{O}_X) \) (for us, locally free \( \mathcal{O}_X \)-module with finite rank at every point). In the case \( (X, \mathcal{O}_X) \) is the affine scheme \( \text{Spec} \ R \), with \( R \) a commutative noetherian ring, then
\[
\text{Vect}(X) = \mathcal{T}_R
\]
since free \( \mathcal{O}_X \)-modules are precisely the projective ones.

**Proposition 3.** For every scheme \( X \) and \( \mathcal{O}_X \)-module \( F \), the following are equivalent:

1. \( F \) is a vector bundle on \( X \).
2. \( F \) is a coherent sheaf and the stalks \( F_x \) are free \( \mathcal{O}_{X,x} \)-modules of finite rank.
3. For every affine open \( U = \text{Spec} \ R \) in \( X \), \( F|_U \) is a finitely-generated projective \( R \)-module.

So we have a dictionary

<table>
<thead>
<tr>
<th>sheaf</th>
<th>( R )-module</th>
</tr>
</thead>
<tbody>
<tr>
<td>coherent sheaf</td>
<td>finitely-generated ( R )-modules</td>
</tr>
<tr>
<td>locally free of finite rank (vector bundles)</td>
<td>finitely-generated projective modules</td>
</tr>
</tbody>
</table>

So we have analogously \( K^0(\text{Vect}(X)) = K_0(R) \) at least in the affine case. In general we have a map
\[
K^0(\text{Vect}(X)) \rightarrow \text{Pic}(X)
\]
by highest exterior powers.
CHAPTER 2

$K_1$ of a ring

Let $R$ be a noetherian ring, not necessarily commutative. We already saw that matrices over $R$ play a role in $K_0(R)$. Let $\mathcal{A}$ be the full subcategory of an abelian category which contains 0, is closed under $\oplus$, and is equivalent to a small category.

Notation: $\mathcal{A}$ as above, $\mathcal{A}[x, x^{-1}]$ is defined to be the following category: objects are pairs $(A, f)$ where $A$ is an object of $\mathcal{A}$ and $f : A \to A$ is an isomorphism. Morphisms are $A \xrightarrow{\varphi} A$, $B \xrightarrow{\psi} B$.

A sequence $0 \to (A', f') \to (A, f) \to (A'', f'') \to 0$ is then exact if and only $0 \to A' \to A \to A'' \to 0$ is exact.

DEFINITION 3. $K_1(\mathcal{A})$ is the abelian group whose generators are objects $[(A, f)] \in \text{ob}(\mathcal{A}[x, x^{-1}])$.

with relations

(a) If $0 \to [(A', f')] \to [(A, f)] \to [(A'', f'')] \to 0$ is exact then $[(A, f)] = [(A', f')] + [(A'', f'')]$.

(b) $[(A, fg)] = [(A, f)] + [(A, g)]$.

Clearly if $\mathcal{B}$ is a category like $\mathcal{A}$ and $T : \mathcal{A} \to \mathcal{B}$ is a functor that preserves exact sequence then we have an induced homomorphism $K_1(\mathcal{A}) \to K_1(\mathcal{B})$.

REMARK. For $C$ a compact Hausdorff topological space $K^1(X) = K^0(SX)$.

DEFINITION 4. If $R$ is a ring, then $K_1(R) := K_1(P_R)$.

If $\varphi : R \to R'$ is a ring homomorphism, then the functor $T(M) = R' \otimes_R M$ preserves exact sequence of projective modules hence we get a functor from the induce morphisms $\varphi_* : K_1(R) \to K_1(R')$.

PROPOSITION 4. Let $R$ be a ring and let $\mathcal{A} = P_R$. Let $\mathcal{F}$ be the category of finitely-generated free $R$-modules. The inclusion $\mathcal{F} \subset P_R$ induces an isomorphism $K_1(\mathcal{F}) \to K_1(R)$. 

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Proof. Clearly by relation (b) we have \([ (A, 1_A) ] = 0 \) for any \( A \in \mathcal{A} \). Now let \([ (P, f) ] \in K_1(R) \), then there is a projective module \( Q \) such that \( P \oplus Q \cong R^n =: F \in \mathcal{F} \). Let \( g : F \to F \) be defined by \( g = (f \oplus 1_Q) \). Define \( j : K_1(R) \to K_1(\mathcal{F}) \) via \([ (P, f) ] \to [(P \oplus Q, f \oplus 1_Q)] \). Then

\[
i_{ij}[(P, f)] = [(F, g)] = [(P, f)] + [(Q, 1_Q)] = [(P, f)].
\]

Note that \( j \) is well-defined since if \( P \oplus Q' = F' \), \( g' = (f \oplus 1_{Q'}) \). To show that \([ (F, g)] = [(F', g')] \) consider

\[
[(P \oplus Q \oplus P' \oplus 1 \oplus 1 \oplus 1)] \in K_1(\mathcal{F})
\]

we have

\[
[(P \oplus Q \oplus P' \oplus 1 \oplus 1 \oplus 1)] = [(P \oplus Q \oplus P', f \oplus 1 \oplus 1 \oplus 1)].
\]

So \([ (P \oplus Q, f \oplus q)] = [(P \oplus Q', f \oplus 1)] \). □

\[GL(n, R)\] be the group of \( n \times n \) matrices over \( R \) that are invertible. The embeddings

\[
GL(n, R) \subset GL(n + 1, R), A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}
\]

give the definition of the infinite general linear group

\[
GL(R) = \lim_{\to n} GL(n, R).
\]

If \( f \in \text{GL}(n, R) \), consider \([ (R^n, f)] \in K_1(\mathcal{F}) \). We get a homomorphism

\[
GL(n, R) \to K_1(\mathcal{F})
\]

and also \( \text{GL}(R) \to K_1(\mathcal{F}) \). Now let the matrices \( E_{ij}(a) = \text{id} + ace_{ij} \) be the elementary matrices \( (i \neq j) \) and \( E(n, R) \subset \text{GL}(n, R) \) be the subgroup generated by elementary matrices

\[
E(R) = \lim_{\to n} E(n, R).
\]

Theorem 0.1 (Whitehead). Let \( R \) be any ring. Then \( E(R) = [\text{GL}(R), \text{GL}(R)] \).

Proof. Observe that \( E_{ij}(a)^{-1} = (I - ace_{ij}) \). To show the converse inclusion for \( A \in \text{GL}(n, R) \),

\[
\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \in E(2n, R).
\]

One can go from the above matrix to \( I_{2n} \) by a series of steps consisting of multiplication by elements of \( E(2n, R) \). Then observe that

\[
\begin{pmatrix} [A, B] & 0 \\ 0 & I \end{pmatrix} \in E(2n, R).
\]

□
By the next result, we have an equivalent definition for $K_1(R)$:

$$K_1(R) \cong \frac{\text{GL}(R)}{E(R)}.$$ 

Note that as $E(R) = [\text{GL}(R), \text{GL}(R)]$, we immediately have that $K_1(R)$ is the abelianization of $\text{GL}(R)$. Note that universal property

$$\text{GL}(R) \longrightarrow K_1(R)$$

for any abelian group $G$.

**Theorem 0.2.** Let $R$ be any ring. Then the map $\text{GL}(R)/E(R) \rightarrow K_1(R) = K_1(\mathcal{F})$ is an isomorphism.

**Proof.** We already know that $\text{GL}(R) \rightarrow K_1(R)$. This gives surjectivity. We construct $j: K_1(\mathcal{F}) \rightarrow \text{GL}(R)/E(R)$ which is an inverse to the map. Let $[(F, f)] \in K_1(\mathcal{F})$ where $F$ is a rank $n$ free module. Take $(e_1, \ldots, e_n)$ to be a basis for $F$. Let $A$ be the corresponding matrix for $f$. Then $j[(F, f)] = [A]$.

If $A'$ is another matrix obtained as above, there is an isomorphism

$$\varphi: (F \oplus F, f \oplus 1) \rightarrow (F \oplus F, 1 \oplus f)$$

in $\mathcal{F}(x, x^{-1})$ making

$$F \oplus F \longrightarrow F \oplus F$$

$$\downarrow$$

$$F \oplus F \longrightarrow F \oplus F$$

commute. Therefore $\left( \begin{array}{cc} A & 0 \\ 0 & I \end{array} \right)$ and $\left( \begin{array}{cc} I & 0 \\ 0 & A \end{array} \right)$ give the same element in $\text{GL}(R)/E(R)$. This shows that $j$ is well-defined.

It also preserves relations of $K_1$; for the relation

$$0 \rightarrow (F', f') \rightarrow (F, f) \rightarrow (F'', f'') \rightarrow 0$$

pick a basis $e'_1, \ldots, e'_n$ for $F'$, extend it to a basis $e'_1, \ldots, e'_n, e''_1, \ldots, e''_m$ for $F$ so that the images of $e''_i$ form a basis for $F''$. If $A', A''$ correspond respectively to $f'$ and $f''$, we get

$$A = \left( \begin{array}{cc} A' & B \\ 0 & A'' \end{array} \right)$$

for the matrix corresponding to $f$. But

$$A = \left( \begin{array}{cc} A' & 0 \\ 0 & I \end{array} \right) \left( \begin{array}{cc} I & 0 \\ 0 & A'' \end{array} \right) \left( \begin{array}{cc} I & B \\ 0 & I \end{array} \right)$$
and therefore \( j[(F, f)] = j[(F', f')] + j[(F'', f'')] \).

\[ \square \]

**Corollary 9.** If \( F \) is a field, then \( K_1(F) \cong F^* \) via the determinant map.

### 1. Stable range results

If \( R \) is commutative then

\[
\det : \text{GL}(R) \to R^*
\]

is split. Let \( SK_1(R) = \ker(\det) \) so \( K_1(R) \cong R^* \oplus SK_1(R) \). Let

\[
\text{SL}(n, R) = \ker(\text{GL}(n, R) \to R^*), \text{SL}(R) = \lim_{\text{lim}} \text{SL}(n, R).
\]

So we have

\[
E(R) \subset \text{SL}(R) \subset \text{GL}(R).
\]

For a general ring \( R \), define

\[
SK_1(R) := \text{Sl}(R)/E(R).
\]

If \( R \) is a field, then \( \text{SL}(n, R) = E(n, R) \), hence \( SK_1(R) = 0 \) and \( K_1(R) \cong R^* \).

Now the questions are:

1. When are the following maps surjective/isomorphisms?
   \[
   \text{GL}(n, R)/E(n, R) \to K_1(R).
   \]

2. Is \( \text{GL}(n, R)/E(n, R) \) a group? an abelian group?

**Definition 5.** The integer \( n \) defines a stable range for \( \text{GL}(R) \) if whenever \( r > n \), and \((a_1, \ldots, a_r)\) is a unimodular row, then there exist \( b_1, \ldots, b_r \in R \) such that

\[
(a_1 + a_r b_1, \ldots, a_{r-1} a_r b_{r-1})
\]

is a unimodular row.

**Lemma 6.** If \( n \) defines a stable range for \( \text{GL}(R) \), and if \( r > n \) and \((a_1, \ldots, a_r)\) is a unimodular row, then there is \( A \in E(r, R) \) such that

\[
(a_1, \ldots, a_r) A = (1, 0, \ldots, 0).
\]

**Theorem 1.1.** If \( n \) defines a stable range for \( R \), then

1. \( \text{Gl}(m, R)/E(m, R) \to \text{GL}(R)/E(R) \) is surjective for all \( m \geq n + 1 \).
2. \( E(m, R) \) is a normal subgroup of \( \text{Gl}(m, R) \), if \( m \geq n + 1 \).
3. \( \text{Gl}(r, R)/E(r, R) \) is an abelian group if \( r \geq 2n \).

**Remark.** Bass, Milnor and Serre, proved that if \( R \) is commutative, the stable range for \( R \) is 1. If \( R \) is the ring of integers of an algebraic number field, then \( st.r(R) = \{0\} \).
Proposition 5. If $D$ is a division ring, then the inclusion $D^* \to GL(1,D) \to GL(D)$ induces a surjection

$$D^*/[D^*,D^*] \to K_1(D).$$

Remark. If $(R,m)$ is a local ring, then the inclusion $R^* \to GL(1,R) \to Gl(R)$ induces a surjection

$$R^*_{ab} \to K_1(R).$$

One can in fact define a non-commutative determinant for the local rings to get an isomorphism

$$E^*_{ab} \to K_1(R).$$

For semi-local rings, again there is a surjection

$$R^*_{ab} \to K_1(R).$$

The idea in defining a non-commutative determinant for local rings is

1. In any $n \times n$ matrix in $Gl(n,R)$, any given row has at least one element which is not in $m$ and is therefore a unit.

2. Use induction to define the usual determinant along with the last step to define determinant.

2. $K_1$ and topology

These are mainly works of Milnor and C. T. C. Wall. Suppose $(K,L)$ is a pair, consisting of a finite connected CW complex $K$ and a subcomplex $L$ which is a deformation retract of $K$. Then

$$G := \pi_1(K) = \pi_1(L)$$

as we know. Milnor defined an element $\tau(K,L)$ called the Whitehead torsion, in the Whitehead group

$$Wh(G) := K_1(\mathbb{Z}[G])/\langle \langle \pm g \rangle \rangle.$$

Whitehead torsion has applications in surgery theory, Poincare conjecture, etc. Milnor showed that the Whitehead torsion of a homotopy equivalence between finite CW-complexes can be used to distinguish between homotopy and simple homotopy types. Some of the computations done where for $G = \mathbb{Z}/5$ when $Wh(G)$ is infinite and if $G \cong \mathbb{Z}/2$ the $Wh(G) = \{1\}$. For $R = k[x,y]/(x^2 + y^2 - 1)$ we get

$$SK_1(R) \neq 0, SK_1(R) \cong \mathbb{Z}/2\mathbb{Z}.$$
3. Description of $K_1$ using group cohomology

Recall the definition of a $G$-module for an arbitrary group $G$.

Example 3.1. If $K$ is a field and $L/K$ is a Galois extension, then the Galois group $G = \text{Gal}(L/K)$ acts on $L^\times$.

If $G$ is any group, $Z$ is the trivial $G$-module. We have definitions

$$H^0(G, M) = M^G = \{ m \in M : g.m = m, \forall g \in G \}$$

and the dual notion is

$$H_0(G, M) = M_G = M/I_G M.$$

Here $I_G$ is the augmentation ideal defined to be the kernel of the augmentation map

$$\mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z}, \text{ via } \sum a_g g \mapsto \sum a_g.$$

As an example $H_0(G, \mathbb{Z}) = \mathbb{Z}$.

Define

$$Z(G, M) = \{ f : G \to M : f(g_1g_2) = g_1f(g_2) + g_2f(g_1) \}$$

to be the set of Crossed homomorphisms. For $m \in M$, $f_m : G \to M$ via $g \mapsto gm - m$ is a crossed homomorphism. Then let

$$B^1(G, M)$$

be the group of boundaries. Then the group cohomology of $M$ is defined as

$$H^1(G, M) = Z(G, M)/B^1(G, M).$$

In particular

$$H^1(G, \mathbb{Z}) = \text{Hom}(G, \mathbb{Z}) = \text{Hom}(G^{ab}, \mathbb{Z}).$$

Once defined systematically, one gets

$$H_1(G, \mathbb{Z}) = H_1(G, \mathbb{Z})^\vee = G^{ab}.$$

A consequence of this is a description

$$K_1(R) = H_1(\text{GL}(R), \mathbb{Z}).$$

4. Baby $K$-theory long exact sequence

**Theorem 4.1 (Resolution theorem).** Suppose $\mathcal{M}$ and $\mathcal{P}$ are categories with exact sequences both contained in an abelian category $\mathcal{A}$ and $\mathcal{P} \subset \mathcal{M}$ is a full subcategory. Assume the following

1. For each object $M$ of $\mathcal{M}$, there is an epimorphism $P \to M$ in $\mathcal{A}$ with $P \in \text{ob}(\mathcal{P})$ such that every endomorphism of $M$ lifts to an endomorphism of $P$.

2. If $\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_0 \to M \to 0$ is exact in $\mathcal{M}$ with $P_i \in \text{ob}(\mathcal{P})$ then $\ker d_n \in \text{ob}(\mathcal{P})$ for $n >> 0$. 


(3) If \( 0 \to M_1 \to M_2 \to M_3 \to 0 \) is a short exact sequence in \( \mathcal{A} \) with \( M_2, M_3 \in \text{ob}(\mathcal{M}) \) then \( M_1 \in \text{ob}(\mathcal{M}) \).

Then the inclusion \( \mathcal{P} \subset \mathcal{M} \) induces an isomorphism \( K_1(\mathcal{P}) \cong K_1(\mathcal{M}) \).

**Proof.** Let \([M, \alpha]\) be an object in \( K_1(\mathcal{M}) \), \( \alpha : M \to M \) an automorphism of \( M \). Consider \( \alpha \oplus \alpha^{-1} \in \text{Aut}(M \oplus M) \). Then \( \alpha \oplus \alpha^{-1} \) is a product of matrices

\[
\begin{pmatrix}
A & 0 \\
0 & A^{-1}
\end{pmatrix} = \begin{pmatrix}
I & A \\
0 & I
\end{pmatrix} \begin{pmatrix}
I & -A^{-1} \\
0 & I
\end{pmatrix} \begin{pmatrix}
0 & I \\
I & 0
\end{pmatrix}
\]

so it can be written as a product of elementary automorphisms of the form

\[
\begin{pmatrix} 1_M & \beta \\ 0 & M \end{pmatrix}, \begin{pmatrix} 1_M & 0 \\ \gamma & 1_M \end{pmatrix}, \quad \beta, \gamma \in \text{End}(M).
\]

Lift \( \beta, \gamma \) to elements in \( \text{End}(P) \) by (1).

\[
\begin{array}{ccc}
P & \longrightarrow & M \\
\beta, \gamma \downarrow & & \downarrow \beta, \gamma \\
P & \longrightarrow & M
\end{array}
\]

Hence we can lift \( \alpha \oplus \alpha^{-1} \) to an automorphism of \( P \oplus P \). Since the kernel is in \( M \) by (3) we keep repeating this process and take the corresponding alternating sums of the resolution obtained in this manner. \( \square \)

We have similar corollaries from this theorem as in case of \( K_0 \):

1. \( K_1(\mathcal{P}_R) \cong K_1(\mathcal{R} - \text{mod}) := G_1(\mathcal{R}) \) if the global dimension of \( \mathcal{R} \) is finite.
2. \( K_1(\mathcal{R}[t]) = K_1(\mathcal{R}) \) if \( \text{gl. dim} \mathcal{R} < \infty \).
3. \( K_1(\mathcal{R}[t, t^{-1}]) = K_1(\mathcal{R}) \oplus K_0(\mathcal{R}) \).

Let \( f : \mathcal{R} \to \mathcal{R}' \) be a homomorphism of rings. This gives a homomorphism \( K_1(\mathcal{R}) \to K_1(\mathcal{R}') \). We define a category \( \mathcal{P}_f \) with objects \((A, g, B)\) where \( A, B \) are projective left \( \mathcal{R} \)-modules and \( g : \mathcal{R}' \otimes \mathcal{R} A \to \mathcal{R}' \otimes \mathcal{R} B \) is an isomorphism. A morphism between \((A, g, B)\) and \((A', g', B')\) is a pair \((\alpha, \beta)\) where \( \alpha : A \to A' \), \( \beta : B \to B' \) such that

\[
\begin{array}{ccc}
\mathcal{R}' \otimes \mathcal{R} A & \xrightarrow{1 \otimes \alpha} & \mathcal{R}' \otimes \mathcal{R} A' \\
\downarrow g & & \downarrow g' \\
\mathcal{R}' \otimes \mathcal{R} B & \xrightarrow{1 \otimes \beta} & \mathcal{R}' \otimes \mathcal{R} B'
\end{array}
\]

is commutative and

\[
0 \to (A', g', B') \to (A, g, B) \to (A'', g'', B'') \to 0
\]

is exact (meaning \( 0 \to A' \to A \to A'' \to 0 \) and \( 0 \to B' \to B \to B'' \to 0 \) are exact).
4. BABY $K$-THEORY LONG EXACT SEQUENCE

**Definition 6.** Let $K_0(R, f)$ be an abelian group defined similar to $K_0$ and $K_1$ with generators $(A, g, B) \in \text{ob}(\mathcal{P}_f)$ with the scissor relations and an extra relation

$$(A, gh, B) = (A, h, C) + (C, g, B).$$

One can check that if we associate to $\theta \in K_1(R')$ the triple $(R^n, \theta, R^n)$ (viewing $\theta$ as an $n \times n$ matrix in $\text{GL}(n, R')$) we get a well-defined mapping

$$K_1(R') \to K_0(R, f).$$

It is straightforward to check that

**Theorem 4.2.** Let $f : R \to R'$ be a homomorphism of rings. Then the sequence

$$K_1(R) \to K_0(R') \to K_0(R, f) \to K_0(R) \to K_0(R')$$

is exact.

**4.1. Localization sequence.** Let $R$ be commutative and $S \subset R$ be a multiplicative closed set and suppose $\text{gl.dim } R < \infty$. Let $S - \text{tor}$ be the subcategory of $R - \text{mod}$ consisting of objects $M \in R - \text{mod}$ such that $S^{-1}M = 0$. $S - \text{tor}$ is closed with respect to exact sequence: If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of $R$-modules, then if $S^{-1}M_i = 0$ for any two for the $i$’s, $S^{-1}M_i = 0$ for the other one. One gets an exact sequence

$$K_0(S - \text{tor}) \to K_0(R - \text{mod}) \to K_0(S^{-1}R - \text{mod}) \to 0$$

called the localization sequence.

**Remark.** If fact $K_0(S - \text{tor}) \cong K_0(R, f)$ for the inclusion mapping $f : R \to S^{-1}R$. 
CHAPTER 3

$K_2$ groups

1. Milnor’s $K_2$

Let $R$ be a ring and $E(R)$ be the set of elementary matrices. $E(R)$ is generated by $E_{ij}(\lambda), \lambda \in R$ with relations

$$E_{ij}(r)E_{ij}(s) = E_{ij}(r+s)$$

(1)

$$[E_{ij}(s), E_{k\ell}(t)] = \begin{cases} E_{i\ell} & j = k, i \neq \ell \\ \text{id} & j \neq k, i \neq \ell \\ E_{ikj}(-ts) & j \neq k, i = \ell \end{cases}$$

(2)

The Steinberg group, $St(R)$, is on the other hand defined as a non-abelian group with generators $x_{ij}(t), t \in R$ where $i \neq j$ ranges over positive integers with the following relations:

$$x_{ij}(s)x_{ij}(t) = x_{ij}(s+t).$$

(1)

$$[x_r(s), x_{k\ell}(t)] = \begin{cases} x_{i\ell}(st) & j = k, i \neq \ell \\ 1 & i \neq \ell, r \neq k \\ x_{kj}(-ts) & j \neq k, i = \ell \end{cases}$$

(2)

We can view $St$ as a functor from the category of rings to the category of groups.

Remark. Since $[a,b]^{-1} = [b,a]$ the relation

$$[x_{ij}(s), x_{ki}(t)] = x_{kj}(-ts)$$

holds. We clearly have a group homomorphism $f : St(R) \to E(R)$.

Generalizing the above notion with define $St_n(R)$ as the quotient of the free group on symbols $x_{ij}^{(n)}(a)$ for $1 \leq i, j \leq n, i \neq j$ and $a \in R$. The relations will be as before. Then we take the direct limit

$$St(R) := \lim_{\rightarrow} St_n(R).$$
Definition 7. $K_2(R)$ is defined to be the kernel of the surjection:

$$K_2(R) := \ker(f : St(R) \to E(R)).$$

Lemma 7. $K_2(R)$ is the center of $St(R)$.

Proof. Let $x \in Z(St(R))$. Then $f(x) \in Z(E(R))$ as $f$ is surjective. So it suffices to show that $Z(E(R))$ is trivial. This is clear since if $A \in \text{GL}(n,R)$ is in the center, $A.E_{ij}(1) = E_{ij}(1).A$ hence $A$ is a diagonal matrix with all entries on the diagonal equal. Therefore $A = \text{id}$.

For the other direction let $\alpha \in K_2(R) \cap \text{im}(St_{n-1}(R) \to St(R))$. So we can write $\alpha$ as a word in $x_{ij}(s)$ with $i,j < n$. Let $A_n$ be the subgroup of $St(R)$:

$$A_n = \{x_{in}(t) : 1 \leq i \leq n-1, t \in R\}.$$

By the relation

$$[x_{ij}(s), x_{k\ell}(r)] = 1, \text{ if } i \neq \ell, j \neq k, r, s \in R$$

we see that $A_n$ is commutative. From the relation $x_{ij}(s)x_{ij}(t) = x_{ij}(s+t)$ we have $x_{ij}(0) = 1$. Hence each element of $A_n$ has a unique expression of the form

$$x_{1n}(a_1)\cdots x_{n-1,n}(a_{n-1}).$$

Thus $f|_{A_n} : A_n \to E(R)$ is an isomorphism onto the group of matrices in $\text{SL}_n(R)$ of the form

$$\begin{pmatrix}
1 & 0 & \cdots & a_1 \\
0 & 1 & \cdots & a_2 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}$$

Further if $i,j < n$ then

$$x_{ij}(a)x_{kn}(b)x_{ij}(-a) = \begin{cases}
x_{kn}(b) & \text{if } j \neq k \\
x_{in}(ab)x_{kn}(b) & \text{if } j = k
\end{cases}.$$

Therefore $\alpha$ normalizes $A_n$. But $f|_{A_n}$ is injective and $f(\alpha) = 1$ as $\alpha \in K_2(R) = \ker(St(R) \to E(R))$. Hence $\alpha$ centralizes $A_n$:

$$[\alpha, x_{in}(a)] = 1, \quad \forall 1 \leq i \leq n-1, a \in R.$$ 

Similarly $[\alpha, x_{ij}(a)] = 1$ for any $j \in \{1, \ldots, n-1\}$ and $a \in R$. Thus $\alpha$ commutes with $x_{ij}(ab)$ for all $a, b \in R$ and $1 \leq j \neq i \leq n-1$. This holds for all large $n$ completing the proof. \qed

2. $K_2$ as an obstruction element

If $G$ acts on $M$, the second group cohomology $H^2(G, M)$ coincides with the (group of ) central extensions:

$$0 \to M \to E \to G \to 0, M \subset Z(E).$$

So we suspect a relation between $K_2(R)$ and central extensions. We will explore this relation in what follows.
**Definition 8.** An extension $1 \to A \to E \to G \to 1$ is called a universal central extension if

1. it is central,
2. given any other central extension $1 \to A' \to E' \to G \to 1$ then there is a unique surjective $\varphi : E \to G$ such that

$$
\begin{array}{ccc}
1 & \to & A \\
& \downarrow & \downarrow \\
& E & \to G
\end{array}
\quad
\begin{array}{ccc}
1 & \to & A' \\
& \downarrow & \downarrow \\
& E' & \to G
\end{array}
$$

commutes.

**Theorem 2.1.** A group $G$ has a universal central extension if and only if $G$ is perfect (i.e. $G = [G, G]$). In this case, a central extension $(E, \varphi)$ is universal if and only if the following two conditions hold:

(i) $E$ is perfect,

(ii) all central extensions of $E$ are trivial.

**Proof.** For necessity suppose $G$ has a nontrivial abelian quotient $A$. Let $\psi : G \to A$ be the quotient map. Then if

$$
1 \to A' \to E \to G \to 1
$$

is a central extension of $G$ we have two distinct morphisms from $(E, \varphi)$ to $(G \times A \to G)$:

$$
\begin{array}{ccc}
E & \to & G \\
(\varphi, 1) & \downarrow & \downarrow \\
G \times A & \to & G
\end{array}
\quad
\begin{array}{ccc}
E & \to & G \\
(\varphi, \psi \circ \varphi) & \downarrow & \downarrow \\
G \times A & \to & G
\end{array}
$$

so $(E, \varphi)$ cannot be universal.

For sufficiency, consider a presentation of $G$ with a free group $F$ and a subgroup $R$ of relations:

$$
1 \to R \to F \to G \to 1.
$$

Step 1. We have a surjection

$(*) \quad \frac{[F, F]}{[F, R]} \to \frac{[F, F]}{R} = \frac{[F/R, F/R]}{[G, G]} = G.$
We will show that this is in fact a central extension of \( G \).

Step 2. Show that any central extension of \( G \) satisfying (i) and (ii) is universal. Let \((E, \varphi)\) be a central extension of \( G \) satisfying (i) and (ii). Let \((E', \varphi')\) be any other central extension of \( G \). If 

\[
\begin{array}{ccc}
E & \xrightarrow{\varphi} & G \\
\downarrow{\psi} & & \downarrow{1} \\
E' & \xrightarrow{\varphi'} & G
\end{array}
\]

are two morphisms, for \( x \in E \) we have \( \varphi' \circ \psi(x) = \varphi' \circ \psi'(x) = \varphi(x) \). So \( \psi(x) = c_x \psi'(x) \) where \( c_x \in A' = \ker \varphi' \). If \( y \in E \), then \( \psi(y) = c_y \psi'(y) \) for \( c_y \in A' = \ker \varphi' \) and \( c_x, c_y \) are central. Hence

\[
\psi([x, y]) = [\psi(x), \psi(y)] = [\psi'(x), \psi'(y)].
\]

Thus \( \psi = \psi' \) on \([E, E]\) but \( E = [E, E] \) showing uniqueness.

We still need to construct a morphism \((E, \varphi) \to (E', \varphi')\) to show existence. Let \( E'' = E \times_G E' \). Since \( \varphi, \varphi' \) are surjective, \( \pi : E'' \to E \) is surjective. Note that \( \ker(\pi_1) \cong A' = \ker \varphi' \). Therefore \( \pi : E'' \to E \) is a central extension of \( E \). Check the remaining details.

Step 3. Show that \((*)\) satisfies (i) and (ii). Let \( E = [F : F', F : R], \varphi : E \to G \). We have a diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\varphi} & F/R = G \\
\downarrow{\psi} & & \downarrow{1} \\
E_1 = F/[F, R] & \xrightarrow{\varphi'} & F/R
\end{array}
\]

So \( \ker \varphi_1 \subset R/[F, R] \) and hence \([\ker \varphi_1, E_1] \subset [R/[F, R], F/[F, R]] \). So \( \ker \varphi_1 \) is central in \( E_1 \) and therefore \( \varphi, \varphi_1 \) are central extensions of \( G \). We check (i) and (ii) for \( E \).

(i) Clearly \( E_1 \) has the property that \([E_1, E_1] = E \). Let \( e_1 \in E_1 \) then there is \( e \in E \) such that \( \varphi(e) = \varphi_1(e_1) \) therefore \( \varphi_1(ee_1^{-1}) = 1 \). Therefore \( ee_1^{-1} \in \ker \varphi_1 \). Hence

\[
E = [E_1, E_1] = [E \ker \varphi_1, E \ker \varphi_1] = [E, E]
\]
as \( \ker \varphi_1 \) is central. We conclude that \( E \) is perfect.

(ii) Let \( 1 \to A \to E_1 \xrightarrow{\psi} E \to 1 \) be any central extension. Define \( E_3 = E_1 \times_G E_2 \). Consider \( \pi_1 : E_3 \to E_1 \). The claim is that \((E_3, \pi_1)\) is a central extension.

\[
\ker(\pi_1) \cong \ker(\varphi \circ \psi) : E_2 \to G.
\]

But \( E = [E, E] \) hence \( \psi([E_2, E_2]) = [\psi(E_2), \psi(E_2)] = [E, E] \). Hence \( E_2 = [E_2, E_2]A \). Also

\[
\psi([E_2, \ker \varphi \circ \psi]) \subset [E, \ker \varphi] = 1
\]
as \( \ker \varphi \) is central. We conclude from all this that if \( x \in \ker \varphi \circ \psi \) then \( x \) commutes with \([E_2, E_2]\) and \( A \) (as \( A \) is central). Therefore \( x \) commutes with all of \( E_2 \). Hence \( E_3 \) is central. Consider

\[
\eta \\
\downarrow \\
E_3 \\
\phi \pi_1 E_1 = F/[F,R]
\]

Use \( \eta \) to get a homomorphism \( \theta : F \to E_2 \) such that if \( x \in F \), then \( \varphi \circ \psi(\theta(x)) \) is the image of \( x \) in \( G = F/R \). Therefore \( \theta \) descends to

\[
\bar{\theta} : F/[F,R] = E_1 \to E_2.
\]

and together with the identity map on \( E_1 \) this gives a splitting \( E_1 \to E_1 \times E_2 = E_3 \). Then we restrict this to \( E \) gives the desired trivialization proving (ii). \( \square \)

**Corollary 10.** \( K_2(R) \) is the kernel of a universal central extension of \( E(R) \).

### 3. Constructing elements of \( K_2 \)

Let \( R \) be any ring and consider two matrices \( A, B \in E(R) \) which commute. Choose representatives \( a, b \in \text{St}(R) \) under \( f : \text{St}(R) \to E(R) \). Then we define

\[
A * B := aba^{-1}b^{-1} \in K_2(R)
\]

which is independent of the liftings as \( K_2R \) is central.

**Lemma 8.** The following statements hold:

1. *Star is skew-symmetric:* \((A * B)^{-1} = B * A\).
2. *Star is bimultiplicative:* \(A_1A_2 * B = (A_1 * B)(A_2 * B)\).
3. \((PAP^{-1}) * (PB P^{-1}) = A * B\).

At least when \( R \) is commutative, given \( u, v \in R^* \) then

\[
D_u = \begin{pmatrix} u & 0 & 0 \\ 0 & u^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D'_v = \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & v^{-1} \end{pmatrix}
\]

then \( D_u \) and \( D'_v \) commute giving \( \{u, v\} = [u, v] \in K_2(R) \).

**Remark.** We haven’t shown that the elements constructed this way are nontrivial. We do not even know yet that \( K_2(R) \) is nontrivial. In case of fields for instance, a highly nontrivial fact is that \( \{-1, -1\} \neq 0 \), and more generally \( \{a, 1 - a\} \neq 0 \).

A central simple algebra is a finite dimensional algebra over its center with no proper nontrivial two sided ideals. Examples are \( M_n(F) \), the quaternion algebra \( \mathbb{H} \), \( M_n(D) \) if \( D \) is a division algebra.
3. CONSTRUCTING ELEMENTS OF $K_2$

**Theorem 3.1.** The Steinberg group $St(R)$ is the universal central extension of $K_2(R)$.

**Proof.** It is easy to see that $St(R)$ is perfect. All we have to do is to construct a section for a central extension

$$1 \to C \to Y \xrightarrow{\varphi} St(R) \to 1.$$ 

For $x_{ij}(a) \in St(R)$, choose an index $k$ distinct from $i$ and $j$ and let

$$y = \varphi^{-1}(x_{ik}(\ell)), y' = \varphi^{-1}(x_{kj}(a)).$$

Then we let $s_{ij}(a) = [y, y']$. Note that $\varphi(s_{ij}(a)) = x_{ij}(a)$. One shall show now that $s_{ij}(a)$ is independent of the choice of index $k$, and satisfies Steinberg relations. For this we consider any central extension

$$1 \to C \to Y \to St(n, R) \to 1(n \geq 0)$$

then for $x, x' \in St(n, R)$ the symbol $[\varphi^{-1}(x), \varphi^{-1}(x')] \in Y$ will denote the commutator $[y, y']$ where $y \in \varphi^{-1}(x)$ and $y' \in \varphi^{-1}(x')$. Using the Steinberg relations one makes the following

**Observation:** If $j \neq k$ and $i \neq \ell$ and $a, b \in R$ then $[\varphi^{-1}(x_{ij}(a)), \varphi^{-1}(x_{kl}(b))] = 1$ in $Y$.

So if we choose four distinct indices $h, i, j, k$ and $u \in \varphi^{-1}(x_{hi}(1)), v \in \varphi^{-1}(x_{ij}(a))$ and $w \in \varphi^{-1}(x_{jk}(b))$ then $[u, v] = 1$. Let $G$ be the subgroup of $Y$ generated by $u, v, w$. Then $G' = [G, G]$ is generated by elements in $\varphi^{-1}(x_{hi}(a)), \varphi^{-1}(x_{ik}(ab))$ and $\varphi^{-1}(x_{hk}(ab))$. This implies that $G'' = 1$ by Steinberg relations. Therefore

$$[\varphi^{-1}x_{hi}(a), \varphi^{-1}x_{jk}(b)] = [\varphi^{-1}x_{hi}(a), \varphi^{-1}x_{ik}(ab)].$$

We set $a = 1$ and conclude

$$s_{hk}(b) = [\varphi^{-1}x_{hi}(1), \varphi^{-1}x_{jk}(b)]$$

i.e. that $s_{ij}$ is independent of the chosen index $h$. Also we have shown that $s_{ij}$ satisfies the first Steinberg relation. For the second relation apply $[u, v][v, w] = [u, vw][v, w, u]$ to $u \in \varphi^{-1}(x_{ij}(1)), v \in \varphi^{-1}(x_{hi}(a)), w \in \varphi^{-1}(x_{jk}(b))$.

$\square$

**Definition 9.** A monomial matrix is one in $GL(n, R)$ that can be expressed as a product $PD$ where $P$ is a permutation matrix and $D$ is a diagonal matrix.

Let $W \subset St(n, R)$ be the subgroup generated by all the $w_{ij}$s. Then and important fact is that if $R$ is commutative, then $\varphi(W) \subset GL(n, R)$ via $\varphi: St(R) \to E(R) \subset GL(R)$. In fact $\varphi(W)$ is at the set of all monomial matrices of determinant 1.

Our goal is to prove

**Theorem 3.2.** If $R$ is a commutative ring, the Steinberg symbols map $R^* \times R^* \xrightarrow{} K_2(R)$ satisfying $\{u, 1-u\} = 1$ for $u \in R^*$ and $1-u \in R^*$.
We define two new symbols:

\[ w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u) \]

\[ h_{ij}(u) = w_{ij}(u)w_{ij}(-1). \]

We see that there are relations such as \([h_{12}(u), h_{13}(v)] = h_{13}(uv)h_{13}(u)^{-1}h_{13}(v)^{-1}\). The following facts are just tedious computations using definitions:

**Lemma 9.** We have \( w_{ij}(u)^{-1} = w_{ij}(-u) \), \( h_{ij}(1) = 1 \) and \( w_{ij}(u) = w_{ji}(-u^{-1}) \). In addition if \( u, v \in R^x \) and \( i \neq j \) and \( k \neq \ell \) then

\[
 w_{k\ell}(u)w_{ij}(v)w_{k\ell}(u)^{-1} = \begin{cases} 
 w_{ij}(-v) & \text{if } j, k, \ell \text{ are distinct} \\
 w_{ij}(-u^{-1}v) & \text{if } k = i \text{ and } j, i, \ell \text{ are distinct} \\
 w_{ij}(-v) & \text{if } k = j \text{ and } j, i, \ell \text{ are distinct} \\
 w_{ji}(-u^{-1}v) & \text{if } k = i \text{ and } j = \ell.
\end{cases}
\]

Take \( u = v \) and we get

\[ w_{k\ell}(u)w_{ij}(u)w_{k\ell}(u)^{-1} = w_{ji}(-u^{-1}). \]

If \( R \) is commutative, \( \{u, v\} = \{v, u\}^{-1} \), \( \{u, u_2, v\} = \{u, v\}\{u_2, v\} \) and also

**Lemma 10.** If \( R \) is a commutative ring and \( u, v \in R^x \) then \( h_{12}(uv) = h_{12}(u)h_{12}(v)\{u, v\}^{-1} \).

**Proof of the theorem.** For \( \{u, -u\} = 1 \) we have to show that

\[ h_{12}(u)h_{12}(-u) = h_{12}(-u^2). \]

Which is the case since the left hand side can be written as

\[ w_{12}(u)w_{12}(-1)w_{12}(-u)w_{12}(-1) = w_{21}(u^{-2})w_{12}(-1) = w_{12}(-u^2)w_{12}(-1) = h_{12}(-u^2). \]

For \( \{u, 1 - u\} = 1 \) we will be showing that

\[ h_{12}(u - u^2) = h_{12}(u)h_{12}(1 - u) \]

and this can be shown starting from the right hand side and rewriting everything in terms of \( w \)'s and \( x \)'s.

Finally we have some nonzero elements

**Proposition 6.**

1. \( \{a, 1\} = 1 = \{1, a\} \).
2. \( \{a^{-1}, b\} = \{a, b^{-1}\} = \{a, b\}^{-1} \).
3. \( \{a, b\} = \{b, a\}^{-1} \).
4. Case of fields

Let $F$ be a field, and let $T(F^x) = \oplus_n T_n(F^x) = \oplus_n (F^x)^{\otimes n}$ be the torus. Then $K^*_M(F) = T(F^x)/I$ where $I$ is the 2-sided ideal generated by all elements of the form $a \otimes (1 - a)$ for all $1 \neq a \in F^x$.

**Theorem 4.1 (Matsumoto).** If $F$ is a field, then $K_2(F)$ has a presentation as the free abelian group on the symbols $\{a, b\}$ with $a, b \in F^x$ and subject to the relations

1. $(a_1a_2,b) = (a_1,b)(a_2,b)$,
2. $(a,b) = (b,a)^{-1}$,
3. $(a,1-a) = 1$.

**Example 4.1 (Symbols on a field).** A symbol on $F$ with values in an abelian group $G$ is a map $(\cdot, \cdot) : F^x \times F^x \to G$ such that it satisfies the above three relations. From Matsumoto’s theorem there is a bijection between symbols on $F$ with values on $G$ and homomorphism from $K_2(F)$ to $G$. One type of tame symbols are the tame symbols for a field $(F, \nu)$ with a discrete valuation $\nu : F^x \to \mathbb{Z}$ on it. Let $\mathcal{O}_\nu$ be the valuation ring $\{x \in F^x : \nu(x) \geq 0\}$ which is a local ring. Then form the natural $\psi : \mathbb{P}^x \to k(\nu)^x$ we have $T_\nu : F^x \times F^x \to k(\nu)^x$ via $(a,b) = \psi((-1)^{\nu(a)+\nu(b)} \frac{a^{\nu(b)}}{\nu(a)}$.

The differential symbols are useful in the context of algebraic geometry. Let $F$ be a field again and $\Omega^1_F = \Omega^1_{F/\mathbb{Z}}$ be the $F$-vector space of Kahler differentials of $F$. As an $F$-vector space $\Omega^1_F$ is spanned by the symbols $da$ for all $a \in F^x$ given via $d(ab) = adb + bda$.

Then $\Omega^2_F = \wedge^2 \Omega^1_F$. The differential symbol $F^x \times F^x \to \Omega^2_F$ is

$$(a,b) \mapsto \frac{da}{a} \wedge \frac{db}{b}.$$ 

Let $L/F$ be a field extension. Then $F^x \to L^x$ induces $L_2(F) \to K_2(L)$. If $L/F$ is finite, then there exists homomorphisms $\text{tr} : K_i(L) \to K_i(F)$ induced by the norm. For instance in the case of $i = 2$ if $(\lambda, \theta) \in K_2(l)$ such that $\lambda \in F, \theta \in L$ then

$$\text{tr}(\lambda, \theta) = (\lambda, N_{L/F}\theta).$$
CHAPTER 4

Galois Cohomology

The setup is this: $G$ is a group. $M$ is a $G$-module, i.e. $M$ is a module over $\mathbb{Z}[G]$. And of course this need not be commutative, so whenever we neglect saying, we assume a left action. We say $M$ is a trivial module if $g.m = m$ for all $g \in G$. For $G$-modules, $M$ and $N$, $\text{Hom}(M, N)$ as groups, has a $G$-module structure via

$$(g.f)(m) = gf(g^{-1}m).$$

On $M \otimes N$ we put also the $G$-module structure $g(m \otimes n) = gm \otimes gn$. If $M$ and $N$ are $G$-modules then

$$\text{Hom}_G(M, N) := \text{Hom}_{\mathbb{Z}[G]}(M, N) = \{ f : M \to N, f(gm) = g.f(m) \}.$$  

Then we defined

$$M^G = H^0(G, M) := \{ m \in M : gm = m, \forall g \in G \} \cong \text{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

where we consider $\mathbb{Z}$ as a trivial $G$-module. The last isomorphism is given by $f \mapsto f(1)$ in the reverse direction. The cohomology groups of $G$ with coefficients in $M$ are a sequence of abelian groups

$$H^q(G, A) \quad q = 0, 1, 2, \ldots$$

such that

1. $H^0(G, M) = M^G$.
2. For $q \geq 0$ the assignment $M \mapsto H^q(G, M)$ is a covariant functor.
3. Given an exact sequence $0 \to M' \to M \to M'' \to 0$ of $G$-module there exists a long exact sequence constructed from connecting homomorphisms

$$\delta_q : H^q(G, M'') \to H^{q+1}(G, M')$$

that fits into a long exact sequence

$$0 \to H^0(G, M') \to H^0(G, M) \to H^0(G, M'') \to H^1(G, M') \to \ldots$$

and that furthermore $\delta$ is functorial.

4. Let $H \subseteq G$ and define $N = \text{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$ (such modules are called co-induced modules from $H$ to $G$). Then $H^q(G, N) = H^q(H, M)$ for all $q \geq 1$.

**Theorem 0.2.** For any group $G$ and any $G$-module $M$, the cohomology groups $H^q(G, M)$ exist for $q \geq 0$ and are unique up to functorial isomorphisms.
The idea is to left-derive $M \to M^G$.

**Proof.** Suppose we have two cohomology theories $H$ and $\bar{H}$. From exact sequence $0 \to M \to M^* \to M' \to 0$ where $M^* = \text{Hom}_\mathbb{Z}(\mathbb{Z}[G], M)$, we have

$$
0 \to M^G \to (M^*)^G \to (M')^G \to H^1(G, M) \to 0
$$

we get $f_1 : H^1(G, M) \to \bar{H}^1(G, M)$ such that $f_1 \delta_0 = \tilde{\delta}_0$. As $\delta_0$ and $\tilde{\delta}_0$ are functorial in $M$ we get that $f_1$ is also functorial. The general proof follows by induction.

For existence we define

$$H^q(G, M) = \text{Ext}^q_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

To compute $\text{Ext}^q_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ we find a $\mathbb{Z}[G]$-projective resolution of $\mathbb{Z}$. Let $P_i$ be the free $\mathbb{Z}[G]$-module $G^{|\mathbb{Z}|i+1}$ given the module structure

$$g \cdot (g_0, \cdots, g_i) = (gg_0, \cdots, gg_i).$$

$P_i$ is a free $\mathbb{Z}[G]$-module with basis $\{(1, g_1, \cdots, g_0) : g_k \in G\}$. The differentials $d_i : P_i \to P_{i-1}$ are defined via

$$(g_0, \cdots, g_i) \mapsto \sum (-1)^i (g_0, \cdots, \tilde{g}_i, \cdots, g_i).$$

This gives an exact sequence of $\mathbb{Z}[G]$-modules $P^* \to \mathbb{Z}$.

For connecting homomorphisms the maps $\delta_i : \text{Hom}_{\mathbb{Z}[G]}(P_i, M) \to \text{Hom}_{\mathbb{Z}[G]}(P_{i+1}, M)$ work:

$$
\delta_i(\theta)(x_1, \cdots, x_{i+1}) = x_1 \delta_1(x_2, \cdots, x_{i+1}) + \cdots + \sum_{1 \leq j \leq i} (-1)^j \theta(x_1, \cdots, x_jx_{j+1}, \cdots, x_{i+1}) + (-1)^{i+1} \theta(x_1, \cdots, x_n).
$$

\[\Box\]

### 1. Lower cohomology groups

So we have expressions

$$H^1(G, M) = B^1(G, M) = \{f_m : m \in M, f_m : G \to M \text{ such that } f_m(g) = gm - m\}.$$ 

For the second cohomology we have $H^2(G, M) = Z^2(G, M)/B^2(G, M)$ where

$$Z^2(G, M) = \{f : G \times G \to M : x_1f(x_2, x_3) - f(x_1, x_2, x_3) + f(x_1, x_2, x_3) - f(x_1, x_2) = 0\}$$

$$B^2(G, M) = \{\delta h : G_1 \times G_1 \to M : h : G \to M \text{ and } \delta h(x_1, x_2) = x_1h(x_2) + h(x_1) - h(x_1x_2)\}.$$ 

If $f$ is a co-cycle, for $(x_1, 1, 1) \in G^3$ we have $xf(1, 1) = f(x, 1)$. The map $f^* : G^2 \to M$ given by $f^*(x_1, x_2) = f(x_1, x_2) - f(x, 1)$ is verified to be a 2-cocycle. In fact $f^* = f - \delta h$ where
2. Computations

Let \( L/K \) be a Galois extension and \( G = \text{Gal}(L, K) \). Then we have the famous

**Theorem 2.1 (Hilbert’s theorem 90).** \( H^1(G, L^\ast) = (e) \).

**Proof.** Let \( f \in Z^1(G, L^\ast) \). By Dedekind’s theorem elements of \( G \) are linearly independent over \( L \). Hence there are elements \( a, b \in L^\ast \) such that

\[
\sum_{\tau \in G} f(\tau) \tau(b) = a.
\]

By cocycle condition we have \( f(\sigma \tau) = \sigma f(\tau) f(\sigma) \) so for any \( \sigma \in G \)

\[
\sigma(a) = \sum_{\tau \in G} \sigma f(\tau) \sigma \tau(b) = \sum_{\tau \in G} f(\sigma \tau) f(\sigma)^{-1} \sigma \tau(b) = a f(\sigma)^{-1}.
\]

Therefore \( ef(\sigma) = \sigma(a^{-1})a \) and hence \( f \) is a coboundary. \( \square \)

**Corollary 11.** Let \( L/K \) be a finite cyclic extension and \( \sigma \) be a generator of \( G \). Let \( a \in L^\ast \) then \( N_{L/K}(a) = 1 \) if and only if there is \( b \in L^\ast \) such that \( a = \sigma b/b \).

**Proof.** If \( a = \sigma b/b \) then

\[
N_{L/K}(a) = a.\sigma a.\ldots.\sigma^{n-1}a = 1
\]

trivially. Conversely suppose \( a \in L^\ast \) with norm 1. Check that the map \( \sigma \mapsto a \) can be extended to a 1-cocyle on \( G \). But by Hilbert’s theorem 90 then this is a coboundary. Hence there is \( b \in L^\ast \) such that \( f(\sigma) = \sigma b/b \). Therefore \( a = (\sigma b)b^{-1} \). \( \square \)

**Proposition 7.** If \( L/K \) is Galois and \( G = \text{Gal}(L/K) \) then for all \( n \geq 1 \) we have \( H^n(G, L) = 0 \).

**Proof.** There is a normal basis for \( L/K \), i.e. there is \( a \in L \) such that \( \{a, \sigma a, \ldots\} \) forms a basis. Any \( b \in L \) can hence be written as

\[
b = \sum b_\sigma \sigma(a).
\]

Then the map \( L \to \text{Hom}_Z(\mathbb{Z}[G], K) \) via \( b \mapsto (\sigma \mapsto b_\sigma^{-1}) \) is an isomorphism of \( G \)-modules (check this). Hence \( L \) is coinduced. Therefore \( H^n(G, L) = 0 \) for all \( n \geq 1 \). \( \square \)
3. Maps in the level of cohomology

3.1. Inflation and restriction. Let $G, G'$ be groups, $M$ a $G$-module and $M'$ is a $G'$-module. Let $f : G' \to G$ be a group homomorphism. Suppose $\varphi : M \to M'$ is a $G'$-homomorphism. Then $(M, M')$ is said to be $(f, \varphi)$-compatible and we get a homomorphism $H^n(G, M) \to H^n(G', M')$ for all $n \geq 0$ from

$$(G')^n \xrightarrow{f^n} G^n \to M \xrightarrow{\varphi} M'.$$

Example 3.1. For $H \to G$ this is called the restriction homomorphism $H^n(G, M) \to H^n(H, M)$.

Example 3.2. If $H \subset G$ is a normal subgroup we have the quotient mapping $G \to G/H$. The action of $G$ on $M$ induces an action of $G/H$ on $M^H$ and we get homomorphisms

$H^n(G/M, M^H) \to H^n(G, M)$.

There are called the inflation maps denoted by $\text{inf}^n$.

The inflation and restriction maps are functorial and commute with the connecting homomorphism.

3.2. Corestriction. Let $M$ be a $G$-module and let $N$ be co-induced as

$N = \text{Hom}_{Z[H]}(Z[G], M).$

Then $H^n(f) : H^n(G, M) \to H^n(G, N) \cong H^n(H, M)$ is the restriction map.

If $H \subset G$ is a subgroup of finite index, let $\{x_i : i \in I\}$ be a set of right coset representatives of $H$ in $G$. We have a $G$-linear map

$\text{Hom}_{Z[H]}(Z[G], M) \to M$

via $f \mapsto \sum_{i \in I} x_i^{-1} f(x_i)$. One can observe that this homomorphism is independent of the choice of representatives. It is moreover functorial and we get induced homomorphisms

$H^n(G, \text{Hom}_{Z[H]}(Z[G], M)) \to H^n(G, M)$

$H^n(H, M)$

called the corestriction map.

The corestriction commutes with the connecting homomorphisms. For $n = 0$ this is just the averaging map

$M^H \to M^G$

$m \mapsto \sum x_i m$. 
Proposition 8. Let $G$ be a group and $H \subset G$ a subgroup. Let $M$ be a $G$-module. The composite
\[ H^n(G, M) \xrightarrow{\text{res}} H^n(H, M) \xrightarrow{\text{cores}} H^n(H, M) \]
is multiplication by $k = [G : H]$.

Example 3.3 (Kummer theory). Let $L/F$ be a Galois field extension with Galois group $\text{Gal}(L/F) = G$. Take $L = \overline{F}$ in characteristic zero and let $G = \text{Gal}(\overline{F}/F)$. Let $\mu_n \subset \overline{F}$ be the Galois module consisting of $n$-th roots of 1. We have
\[ 0 \to \mu_n \to \overline{F}^\times \to \overline{F}^\times \to 1 \]
where the surjection is $x \mapsto x^n$. So
\[ 0 \to H^0(G, \mu_n) \to H^0(G, \overline{F}^\times) \to H^0(G, \overline{F}^\times) \to H^1(G, \mu_n) \to H^1(F, \overline{F}^\times) = 0 \]
is the result of taking the long exact sequence. So we get
\[ 0 \to \mu_n(F) \to F^\times \xrightarrow{n} F^\times \to F^\times/(F^\times)^n. \]
Taking $n = 2$ we have
\[ H^1(G_F, \mu_2) = F^\times/(F^\times)^2. \]

3.3. Cup product. Suppose $G$ is a group and $M, N$ are two $G$-modules. The $M \otimes_Z N$ is again a $G$-module. Then for all $p, q \geq 0$ there exists unique homomorphisms
\[ \cup : H^p(G, M) \otimes H^q(G, N) \to H^{p+q}(G, M \otimes N). \]
such that

(1) are functorial in the sense that
\[ H^p(G, M) \otimes H^q(G, N) \xrightarrow{f \otimes g} H^p(G, M') \otimes H^q(G, N') \xrightarrow{\cup} H^{p+q}(G, M' \otimes N') \]
is commutative.

(2) commutes with boundary, restriction, corestriction and inflation maps.

In fact explicitly the cup product can be expressed as follows: if $a : G^p \to M$ and $b : G^q \to N$ are elements are $H^p(G, M)$ and $H^q(G, N)$ respectively then
\[ (a \cup b)(g_1, \ldots, g_{p+q}) = a(g_1, \ldots, g_p) \otimes b(g_{p+1}, \ldots, g_{p+q}). \]
Example 3.4. Recall that \( H^1(G_F, \mu_n) = F^*/(F^*)^n \). The special elements in \( H^2(G_F, \mu_2) \) are of the form

\[
\{ \lambda_1, \lambda_2 \}, \quad \lambda_i \in F^*/(F^*)^2.
\]

These are called symbols in Galois cohomology. We have

\[
K^n_1(F) = F^* = H^0(G_F, F^*).
\]

Thus we get a map \( K^n_1(F) \to H^1(G_F, \mu_2) \) which is a homomorphism: \( \lambda \mapsto [\lambda] \in F^*/(F^*)^2 \).

It follows that since \( K^n_1(F) \) is generated by symbols \( \{\lambda_1, \ldots, \lambda_n\} \) then \( H^n(G_F, \mu_2) \) is generated by

\[
[\lambda_1] \cup \cdots \cup [\lambda_n].
\]

There is a theorem of Merkurjev showing that there is an isomorphism \( K^2_2(F) \to H^2(G, \mu_2) \) induced as such and called the norm residue homomorphism.

Proposition 9. Let \( H \subset G \) be a normal subgroup and \( A \) a \( G \)-module. If \( H^i(H, A) = 0 \) for \( 1 \leq i \leq n-1 \) (no condition if \( n = 1 \)) then the sequence

\[
0 \to H^n(G/H, A^H) \xrightarrow{\inf} H^n(G, A) \xrightarrow{\res} H^n(H, A)
\]

is exact.

Proof. By induction on \( n \). Let \( n = 1 \) and let \( [f] \in H^1(G/H, A^H) \) such that \( \inf[f] = 0 \). Then the cocyle \( \inf f \) becomes a coboundary, i.e. there is \( a \in A \) such that \( f(\sigma) = \sigma a - a \) for all \( \sigma \in G \). Since \( f|_H = 0 \) then \( a \in A^h \) then \( f \in B^1(G/H, A^H) \) and therefore the inflation is injective.

The composition \( \res \circ \inf = 0 \). since \( H \to G \to G/H \) is the zero map. Let \( f \in Z^1(G, A) \) be such that \( \res(f) = 0 \). Then \( f(\sigma) = \sigma a - a \) for all \( \sigma \in H \) and \( a \in A \). Let \( f_a : G \to A \) be \( f_a(g) = ga - a \). Thus \( [f] \) and \([f']\) are equal in \( H^1(G, A) \) where \( f' = f - f_a \). But \( f'|_H = 0 \) and hence \( f' \) induces a 1-cocyle: \( f'' : G/H \to A^H \). Then

\[
\inf[f''] = [f]
\]

proving the exactness at \( H^1(G, A) \). Suppose \( n > 1 \) and assume the proposition is true for \( n-1 \). Consider the exact sequence

\[
0 \to A \to A^x \to A' \to 0
\]

where \( A^x = \text{Hom}_Z(\mathbb{Z}[G], A) \). Taking the long exact sequence we get the short exact sequence

\[
0 \to A^H \to (A^x)^H \to (A')^H \to 0
\]

of \( G/H \)-modules.

Now \( A^x \) is coinduced also as an \( A \)-module and \( (A^x)^H \) is coinduced as a \( G/H \)-module. Taking the long exact sequence associated to the above short exact sequence and comparing it to that of the quotient \( 0 \to G \to H \to G/H \to 0 \) the isomorphism for \( n \geq 2 \) are all proved. \( \square \)
Corollary 12. Let $A = L^\times$ and $G = \text{Gal}(L/K)$ be finite. Let $H \subseteq G$ be a normal subgroup. We get

$$0 \to H^2(G/H, F^\times) \xrightarrow{\inf} H^2(G, L^\times) \xrightarrow{\text{res}} H^2(H, L^\times)$$

is exact.

Remark. Just as in more general setups we have a Hochschild-Serre spectral sequence with page two

$$E_2^{p,q} = H^p(G/H, H^q(H, A))$$

which converges to $H^n(G, A)$. The construction is the same as any Grothendieck spectral sequence.

4. Profinite groups

Let $K$ be a field and $L$ a Galois extension, not necessarily finite. Let $G = \text{Gal}(L/K)$. In this case $G$ comes with a topology called the pro-finite topology. The opens are the normal subgroups corresponding to subfields of $L$ that are finite and Galois over $K$. The fact that

$$L = \bigcup K_\alpha = \varprojlim K_\alpha$$

where $K_\alpha$ ranges over all intermediate field extensions is equivalence to

$$G = \varprojlim G/G_\alpha.$$  

Also note that $G$ is a profinite group in the sense that it is the inverse limit of an inverse system of finite groups. Then

$$G \subset \prod G/G_\alpha$$

is a compact, totally disconnected, Hausdorff topological group if $G_\alpha$’s have discrete topology.

Definition 10. Let $G$ be a profinite group and $A$ a $G$-module. Then $A$ is a discrete $G$-module if $G \times A \to A$ is continuous when $A$ is given the discrete topology.

This is equivalently the case when $A = \bigcup A^{G'}$ for $G'$ ranging over open subgroups of $G$.

$$H^n_c(G, A) = H^n_c(\varprojlim G/H_\alpha, A) = \varprojlim H^n(G/H_\alpha, A) = \varprojlim H^n(G/H_\alpha, A^{H_\alpha}).$$