Motivic Integrations

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CHAPTER 1

Jet spaces and cylindrical sets; the values of the motivic measure

Guillermo Mantilla-Soler, Reference [1]

Basic idea: Let k be algebraically closed field of characteristic zero. Let X/k be a scheme of finite type. The goal of construction of jet spaces has been to study singularities of X using it. It was constructed in '68 by Nash. Let $J_1(X)$ be the germs of regular linear functions on X and $J_2(X)$ germs of quadratic functions and so on. Then

$$J_\infty(X) = \varprojlim J_n(X).$$

An m-jet is an infinitesimal curve on X:

$$\theta : \mathcal{S}pec(k[t]/t^{m+1}) \to X$$

and the set of m-jets has a structure of scheme $J_m(X)$.

We define the functor:

$$\mathcal{F}_{X,m} : f.t.\ \mathcal{S}ch/k^{op} \to \mathcal{S}et$$

via $Z \mapsto \mathrm{Hom}(Z \times \mathcal{S}pec(k[t]/t^{m+1}, X)$ and have

Proposition 1. *$\mathcal{F}_{X,m}$ is representable. Represented by $J_m(X)$.*

Proof. In the affine case need the coordinate ring $C_m(X)$ to satisfy

$$\mathrm{Hom}(\mathbb{C}[x_1, \cdots, x_n]/\langle f_1, \cdots, f_j\rangle, A[t]/t^{m+1}) = \mathrm{Hom}(C_m(X), A)$$

and this is obvious. □

Example 0.1. $J_m(\mathbb{A}^n) = \mathbb{A}^{n(m+1)}$. ⌟

The truncation maps $k[t]/t^{m+1} \to k[t]/t^{m'+1}$ for any $m' \leq m$ induce maps

$$\pi^m_{m'} : J_m(X) \to J_{m'}(X)$$

In particular

$$J_\infty(X)(k) = X(k[\![t]\!]).$$

Proposition 2. *$X \to Y$ etale implies $J_m(X) \cong J_m(Y) \times_Y X$.*

Proof. Analogue to Hensel's lemma. □

If X is smooth of dimension n then $J_m(X)$ is locally an $\mathbb{A}^{m+1}$-bundle over X, in fact $J_m(X) \cong \mathbb{A}^{(m+1)n} \times_{\mathbb{A}^n} X$.

Definition 1. $A \subsetneq J_\infty(X)$ is stable if for $m >> 0$ $A_m := \pi_m(A)$ is a constructible subset of $J_m(X)$ and $\pi_m^{m+1} : A_{m+1} \to A_m$ is locally trivial $\mathbb{A}^n$-bundle.

Definition 2. $A \subseteq J_\infty(X)$ is a cylinder if $A = \pi_m^{-1}(B)$ for some m and $B \subset J_m(X)$ is constructible.

In the smooth case cylinders are stable.

Let $K_0(Var_k)$ be the Grothendieck ring of Var_k. Denote $[\mathbb{A}^1] = \mathbb{L}$ and note that $[\mathcal{S}pec(k)] = 1$. There is a linear mapping

$$\dim : K_0(Var_k) \to \mathbb{Z} \cup \{-\infty\}$$

with $\dim(\tau) \leq d$, if $\tau = \sum_{i=1}^{\infty} a_i[X_i]$ and if $\dim(X_i) \leq d$ for all i and $\dim(\tau) = d$ if $\dim(\tau) \leq d$ and $\dim(\tau) \nleq d - 1$. Then

$$\dim(\tau.\tau') \leq \dim(\tau) + \dim(\tau')$$
$$\dim(\tau + \tau') \leq \max\{\dim(\tau), \dim(\tau')\}$$
$$\dim(\tau) = -\infty \Leftrightarrow \tau = [\varnothing]$$
$$\dim[X] = \dim(X)$$

Whether $\mathbb{L}$ is a zero divisor is not known. (So it is not know whether $M_k = K_0(Var_k)_{\mathbb{L}}$ is nontrivial.) But it is easy to see that $\mathbb{L}$ is not nilpotent hence it makes sense to localize to M_k. The mapping above factors as

$$\dim : M_k \to \mathbb{Z} \cup \{-\infty\}$$

and $\dim(\mathbb{L}^{-1}) = -1$.

Pierrie from Model Theory $\Rightarrow M_k[(1 - \mathbb{L}^{-n})^{-1}]$ is complete.

Then $\mu_k(A) = [A_m]\mathbb{L}^{-mn}$ for stable A defines the measure. Note that $[A_{m+1}] = [A_m]\mathbb{L}^n$.

CHAPTER 2

The motivic measure and change of variables formula

Robert Klinzmann, Reference [1]

Recall that $J_k(Y) \to Y$ is a morphism with

$$\text{Fiber over } y = \{\gamma_k : \mathcal{S}pec\,\mathbb{C}[z]/\langle z^{k+1}\rangle \to Y\}.$$

$A \subseteq J_\infty(Y)$ is cylindric iff for $\pi_k : J_\infty(Y) \to J_k(Y)$ is the truncation map $A = \pi_k^{-1}(B_k)$ for $B_k \subseteq J_k(Y)$ is constructible. We define

$$\widetilde{\mu}(\pi_k^{-1}(B_k)) := [B_k]\mathbb{L}^{-n(k+1)}$$

and have

$$\widetilde{\mu}(\coprod_{i=1}^{\ell} c_i) := \sum_{i=1}^{\ell} \widetilde{\mu}(c_i).$$

Problem: Want to integrate

$$F_D : J_\infty(Y) \to \mathbb{Z}_{\leq 0} \cup \infty$$

but the level set $F_D^{-1}(\infty)$ is not a finite disjoint union of cylinders. But

$$F_D^{-1}(\infty) = \cap_{i\in\mathbb{Z}_{\geq 0}} \pi_k^{-1}(\pi_k(F_D^{-1}(\infty))).$$

Thus

$$J_\infty(Y)\smallsetminus F_D^{-1}(\infty) = J_\infty(Y)\smallsetminus\pi_0^{-1}(\pi_0(F_D^{-1}(\infty)))\coprod\coprod_{k\in\mathbb{Z}_{\geq 0}}(\pi)_k^{-1}(\pi_k(F_D^{-1}(\infty)))\smallsetminus\pi_{k+1}^{-1}(\pi_{k+1})F_D^{-1}(\infty)).$$

So we should go to completion of $M_k = K_0(Var_{\mathbb{C}})[\mathbb{L}^{-1}]$.

Question: What function we want to integrate? Let Y be a smooth variety, D effective divisor on Y with only simple normal crossing. The we want to consider the function

$$F_D : J_\infty(Y) \to \mathbb{Z}_{\geq 0} \cup \infty$$

via

$$F_D(\gamma) := \gamma_u.D = \text{ vanishing order of } g(\gamma_u(Z)) \text{ at } Z = 0\ .$$

Recall that a simple normal crossing is one taht for any $y \in Y$ there is $U \supset y$ such that D is of the form $g := z_1^{a_1}\cdots z_{j_n}^{a_{j_n}}$ for $j_y \leq n$ where $z_1, \cdots, z_n$ coordinates in U.

PROPOSITION 3. *With this setting $F_D^{-1}(s)$ is a cylinder for any $s \in \mathbb{Z}_{\geq 0}$.*

Stratification of $J_\infty(Y)$: Let $D = \sum_{i=1}^r a_i D_i$ and $J \subseteq \{1, \cdots, r\}$

$$D_J =: \begin{cases} \cap_{j\in J} D_j & J \neq \varnothing \\ Y & J = \varnothing \end{cases}$$

and

$$D_J^\circ = D_J \setminus \bigcup_{i\in\{1,\cdots,r\}\setminus J} D_i.$$

Thus

$$Y = \coprod_{J\subseteq\{1,\cdots,r\}} D_J^\circ$$

and

$$J_\infty(Y) = \coprod_{J\subseteq\{1,\cdots,r\}} \pi_0^{-1}(D_J^\circ).$$

Then the stratification of $J_\infty(Y)$ is given via

$$M_{J,S} = \{(m_1, \cdots, m_r) \in \mathbb{Z}_{\geq 0}^r : \sum a_i m_i = s : m_j > 0 \text{ iff } j \in J\}$$

amd

$$\gamma_u \in \pi_0^{-1}(D_J^\circ) \cap F_D^{-1}(s) \Leftrightarrow (D_{D_1}(\gamma_u), \cdots, F_{D_r}(\gamma_u)) \in M_{J,S}.$$

PROPOSITION 4. *F_D is $\widetilde{\mu}$-measurable.*

1. The motivic integral

Let Y be smooth variety of dimension n, and have effective division $D = \sum_{i=1}^r a_i D_i$ which is simple normal crossig.

DEFINITION 3.

$$\int_{J_\infty(Y)} F_D d\mu = \sum_{s\in\mathbb{Z}_{\geq 0}\cup\{\infty\}} \mu(F_D^{-1}(s)).\mathbb{L}^{-s}$$

Want to compute this integral with respect to the stratification above.

THEOREM 1.1.

$$\int_{J_\infty(Y)} F_D d\mu = \sum_{\varnothing\neq J\subseteq\{1,\cdots,r\}} [D_J^\circ](\prod_{j\in J} \frac{\mathbb{L}-1}{\mathbb{L}^{a_j+1}-1})\mathbb{L}^{-n}$$

PROOF.

$$F_D^{-1}(s) = \coprod_J \coprod_{M_{J,s}} \cap_{i=1}^\infty F_{D_i}^{-1}(m_i)$$

Cover $Y = \cup U$. One can show

$$\cap D_{D_i}^{-1}(m_i) \cap \pi_0^{-1}(U)$$

is a cylinder set of the form

$$\prod_t^{-1}((U \cap D_J^\circ) \times \mathbb{C}^{tn - \sum_{i\in J} m_j} \times (\mathbb{C}^*)^{|J|})$$

then do the calculation! □

2. Transformation rule

Let $\alpha : Y' \to Y$ be a proper birational morphism between smooth varieties. Let $W = K_{Y'} - \alpha^* K_Y$.

THEOREM 2.1.

$$\int_{J_\infty(Y)} F_D d\mu = \int_{J_\infty(Y')} F_{\alpha^* D + W} d\mu$$

THEOREM 2.2. *Let $\varphi_i : Y_i \to X$ be resolutions of X with discrepancy divisors D_i, then*

$$\int_{J_\infty(Y_1)} F_{D_1} d\mu = \int_{J_\infty(Y_2)} F_{D_2} d\mu.$$

PROOF. Make Y_0, $\varphi_0 : Y_0 \to X$ be the Hironaka hat of Y_1 and Y_2. Then computing the integral over Y_0 for

$$\begin{array}{ccc} Y_0 & \xrightarrow{\psi_2} & Y_2 \\ {\scriptstyle \psi_1}\downarrow & & \downarrow{\scriptstyle \varphi_2} \\ Y_1 & \xrightarrow[\varphi_1]{} & X \end{array}$$

$$\begin{aligned} K_{Y_0} &= \pi_0^*(K_X) + D_0 = \psi_i^* \pi_i^*(K_X) + D_0 \\ &= \psi_i^*(K_{Y_i}) + (D_0 - \psi_i^*(D_i)) \end{aligned}$$

and from this

$$\int_{J_\infty(Y_i)} F_{D_i} d\mu = \int_{J_\infty(Y_0)} F_{D_0} d\mu.$$

□

REMARK. If $D = \varnothing$ get $\int_{J_\infty(Y)} F_D d\mu = \int_{J_\infty(Y)} 1 d\mu = [Y] \in K_0(Var_{\mathbb{C}})[\mathbb{L}^{-1}]$.

CHAPTER 3

p-adic numbers and measures

Lance Robson

Recall the evaluation on $\mathbb{Z}_p$ the ring of p-adic numbers given by $v(p^n u) = n$, where $n \geq 0$ and $u \in \mathbb{Z}_p^\times$ and $v_p(0) = +\infty$.

Analogy between the power series over $\mathbb{C}$, $\mathbb{C}[[t]]$ and p-adic numbers $\mathbb{Z}_p$: The residue fields are $\mathbb{C} = \mathbb{C}[t]/(t)$ and $\mathrm{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$. The solutions to $f = 0$ for $f \in \mathbb{C}[x_1, \cdots, x_m]$ in $(\mathbb{C}[t]/(t^{n+1}))^m$ correspond to solutions to $0 = f \in \mathbb{Z}[x_1, \cdots, x_m]$ in $(\mathbb{Z}/p^{m+1}\mathbb{Z})^m$ and likewise for solutions in $\mathbb{C}[[t]]^m$ and $(\mathbb{Z}_p)^m$.

LEMMA 1 (Hensel's). *If $f \in \mathbb{Z}_p[x_1, \cdots, x_m]$ and $x \in (\mathbb{Z}_p)^m$ is a solution to $f(x) \cong 0 \mod p$, such that $\partial f(x)/\partial x_j \neq 0$ for some j, then there exists $y \in (\mathbb{Z}_p)^m$ with $f(y) = 0$ and $y_i \cong x_i \mod p$ for each i.*

LEMMA 2 (Hensel's complex version). *If $f \in \mathbb{C}[x_1, \cdots, x_m]$, and $x \in (\mathbb{C}[[t]])^m$ is such that $f(x) \cong 0 \mod t$ and $\partial f(x)/\partial x_j \neq 0$, there there exists $y \in (\mathbb{C}[[t]])^m$ with $f(y) = 0$ and $y_i \cong x_i \mod t$ for each i.*

A field is Henselian if the property in Hensel's lemma is satisfied so $\mathbb{Z}_p$ and $\mathbb{C}[[t]]$ are Henselians.

Haar measure on $\mathbb{Z}_p$: There is a translation invariant regular Borel measure μ on $\mathbb{Z}_p$ where

$$\mu(\mathbb{Z}_p) = 1$$
$$\mu(p^n\mathbb{Z}_p) = p^{-n}$$
$$\mu(x) = \mu(p^n\mathbb{Z}_p)$$

where $x \in \mathbb{Z}_p/p^n\mathbb{Z}_p$ and note that $\mathbb{Z}_p = \bigcup_{x\in\mathbb{Z}_p/p^n\mathbb{Z}_p} x + p^n\mathbb{Z}_p$. Note that any $x \in \mathbb{Z}_p$ has a local basis of neighborhoods $x + p^n\mathbb{Z}_p$. And this topology is induced from the ultra-metric properties

$$v_p(x+y) \geq \min(v_p(x), v_p(y)), \quad |x+y|_o \leq \max\{|x|_p, |y|_y\}$$

of

$$|x|_p = p^{-v_p(x)}.$$

Recall also that $\mathbb{Z}_p$ is totally-disconnected with this topology.

A translation invariant measure on $\mathbb{A}^n(\mathbb{Z}_p)$ is given by product measure construction with μ. Then for instance the fibers of

$$\mathbb{A}^n(\mathbb{Z}_p) \to \mathbb{Z}^n(\mathbb{Z}_p)$$

have volume p^{-n}.

Let X be a smooth scheme over $\mathcal{S}pec\,\mathbb{Z}_p$ of dimension d with a nowhere zero global differential form on X of degree d. Since X is smooth $X(\mathbb{Z}_p)$ is a p-adic manifold. Let $x \in X(\mathbb{Z}_p)$ with $t_1, \cdots, t_d$ local coordinates on a chart $U \to \Theta(U) \subseteq \mathbb{A}^d(\mathbb{Z}_p)$. For some nowhere-zero p-adic analytic function $\varphi : \Theta(U) \to \mathbb{Z}_p$ the form

$$\omega = \Theta^*(\varphi(t_1, \cdots, t_d) dt_1 \wedge \cdots \wedge dt_d)$$

gives μ_ω on U $d\mu_\omega = |\varphi(t)|_p dt$.

Theorem 0.3 (Weil). *Let X be a smooth scheme over $\mathcal{S}pec\,\mathbb{Z}_p$ of dimension d. Then*

$$\int_{X(\mathbb{Z}_p)} d\mu = \frac{|X(\mathbb{F}_p)|}{p^d}.$$

Let $X = \mathbb{A}^N(\mathbb{Z}_p)$ and $Y \subseteq X$ a closed smooth subvariety of X of dimension d everywhere. Then

$$X_n = (\mathbb{Z}/p^n)^N, \quad X = \varprojlim X_n$$

and if we let Y_n is the image of Y in X_n and then $Y = \varprojlim Y_n$. So already $|Y_n| \le |X_n| = p^{nN}$. But due to Serre we in fact have,

Theorem 0.4 (Serre). $|Y_n| = \mu(Y) p^{nd}$ *for n large enough.*

CHAPTER 4

Igusa Zeta functions and monodromy conjecture

Andrew Morrison, Reference [7]

1. Outline story

Starting with $f \in \mathbb{Z}[x_1, \cdots, x_d]$ we have

Number theory	Geometry
char p	char 0
Congruences mod p^n	Singularities of $\{f = 0\}/\mathbb{C}$
Nilpotent extensions along	Nilpotent extensions along
$\mathcal{F}_p \leftarrow \mathbb{Z}/p^2 \rightarrow \mathbb{Z}/p^3 \leftarrow \cdots \leftarrow \mathbb{Z}_p$	$\mathbb{C} \leftarrow \mathbb{C}[\varepsilon]/\varepsilon^2 \rightarrow \mathbb{C}[\varepsilon]/\varepsilon^3 \leftarrow \cdots \leftarrow \mathbb{C}[\![\varepsilon]\!]$
Count solutions, $N_{p,n} =$	Compute arc spaces, $\mathcal{L}_n(f) =$
$\#\{a : f(a_0 + pa_1 + \cdots + p^n a_n = 0 \mod p^{n+1}\}$	$[\{a : f(a_0 + a_1\varepsilon + \cdots + a_n\varepsilon^n) = 0 \mod \varepsilon^{n+1}\}]$
Generating series ⇒p-adic zeta function	Generating series ⇒motivic zeta function

Questions:

(1) Are zeta functions homolophic/rational?

(2) Can we compute them?

(3) Is motivic zzeta function *equal to* the p-adic zeta function?

(4) How do we find their poles?

The first three are answered by motivic/p-adic integration. The last has a conjectural answer, which is the monodromy conjecture.

Remark. In the smooth case (where $\{f = 0\}$ is smooth), $N_{p,n}(f) = N_{p,n-1}(f)p^{d-1}$ by Hensels lemma, whereas

$$[\mathcal{L}_n(f)] = [\mathcal{L}_{n-1}(f)].\mathbb{L}^{d-1}$$

by etale liftings. So the zeta functions are the same and computable.

Remark. The interesting case is when $\{f = 0\}$ is singular. A standard example may be

$$f(x, y) = y^2 - x^3 \in \mathbb{Z}[x, y]$$

.

REMARK. Can generalize to $f \in \mathcal{O}_L[x_1, \cdots, x_n]$, the ring of integers of a number field.

2. p-adic Igasa zeta functions

Recall:

(1) That there is a valuation on $\mathbb{Q}_p$, $\nu_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ via $a_n p^{-n} + \cdots \mapsto -n$ and a norm $|x|_p = \frac{1}{p^{\nu_p(x)}}$ which has ultra-metric properties, $\mathbb{Q}_p$ is complete with respect to it and $\mathbb{Z}_p$ is a unit ball.

(2) And that there is a Haar measure such that $\mu(\mathbb{Z}_p) = 1$ and is translation invariant.

There are also extra structures that are redundant in the naive cases:

(1) Angular component: $ac_p : \mathbb{Q}_p^\times \to \mathbb{Z}_p^\times$ via $x \mapsto xp^{-\nu_p(x)}$ and we set $ac_p(0) = 0$

(2) Character: $\chi : \mathbb{Z}_p^\times \to \mathbb{C}^*$ and we set $\chi(0) = 0$.

(3) A function $\Phi : \mathbb{Q}_p^d \to \mathbb{C}$ which is locally constant and has compact support.

DEFINITION 4. The p-adic Igasa zeta function is

$$Z(f, \chi, \Phi; s) = \int_{\mathbb{Q}_p^d} \Phi.(\chi \circ ac)(f).|f|_p^s d\mu$$

where $s \in \mathbb{C}$ is any complex number with positive real part, Re $(s) > 0$.

REMARK. A naive case is when χ is trivial $\mathbf{1}_{\mathbb{Z}_p^d}$ then we denote it as

$$Z(f; s) := \int_{\mathbb{Z}_p^d} |f|_p^s d\mu$$

DEFINITION 5 (p-adic cylinders). For any $m \geq 0$ and $\pi_m : (\mathbb{Z}_p)^d \twoheadrightarrow (\mathbb{Z}/p^{n+1})^d$ we call $C = \pi_{m_0}^{-1}(S)$ a cylinder of $S \subset (\mathbb{Z}_{p^{n+1}})^d$.

LEMMA 3. $(p^{-d(m+1)}|\pi_m(C)|)_{m \geq 0}$ *is constant as* $m >> 0$ *with limit equal Haal measure.*

PROOF. Let C be a cylinder over $\pi_{m_0}(C)$ and $m \geq m_0$. Then

$$C = \coprod_{a \in \pi_m(C)} (a + (p^{m+1}\mathbb{Z}_p)^d)$$

is a union of disjoint balls. By translation invariance

$$\mu(C) = \sum_{a \in \pi_m(C)} |p^{m+1}\mathbb{Z}_p|_p^d = |\pi_m(C)| p^{-(m+1)d}.$$

□

3. Naive zeta functions and solutions mod p^{n+1}

DEFINITION 6 (Solutions).

$$N_{p,n} = \#\{a \in (\mathbb{Z}/p^{n+1})^d : f(a_0 + a_1 p + \cdots + a_n p^n) = 0 \mod p^{n+1}\}$$
$$J_p(f;t) = \sum_{n \geq 0} N_{p,n}(f) t^n \in \mathbb{Z}[\![t]\!] s$$

PROPOSITION 5. *Set $t = p^{-s}$ then*

$$J_p(f, p^{-d}t) = \frac{t}{1-t}(1 - Z_p(f;s)).$$

PROOF. By definition the coefficient of $t^i = p^{-is}$ in $Z(f;s)$ is the Haar masure $\int |f|_p^s d\mu$. So set

$$S_i = \{z \in \mathbb{Z}_p^d : \nu_p(f(z)) = i\}.$$

The coefficient of the right hand side, $(1 - \sum_{i=0}^{m-1} \mu(S_i))t^m$ is equal to the measure of

$$S_{\geq m} = \{z \in (\mathbb{Z}_p)^d : \nu_p(f(z)) \geq m\}.$$

Now $S_{\geq m}$ is a cylinder over

$$C_{m-1} = \{z \in (\mathbb{Z}_p)^d : f(z) = 0 \mod p^{m-1}\}.$$

By lemma $\mu(S_{\geq m}) = p^{-dm} N_{p,m-1}(f)$ which is the coefficient of the left hand side. □

Example 3.1. Let $f = y^2 - x^3$. To compute the zeta function we count the solutions. We use the notation

$$\overline{ac}_p(x) = x p^{-\nu_p(x)} \mod p.$$

Fix value of y in $\mathbb{Z}/p^m$ and call it a. Solve $x^3 = a^2 \mod p^m$. If $p \neq 3$ the solutions exist when

(1) $2\nu_p(a) \geq m$;

(2) $2\nu_p(a) < m, 2|\nu_p(a)$ and $\overline{ac}_p(a^2) \in \mathbb{F}_p$ is a cube.

In the first case any x such that $e\nu_p(x) \geq m$ is a solution. In the second case let $w = \nu_p(a)/3$. If $b \in \mathbb{F}_p$ such that $b^3 - \overline{ac}_p(a^2) = 0 \mod p$ there is a lift $\widehat{b} \in \mathbb{Z}/p^m$ such that $\overline{ac}_p(\widehat{b}) = b$ where $\widehat{b} - a^2 = 0 \mod p^m$. All such solutions are given by cosets $\widehat{b} + p^{m-4w}\mathbb{Z}/p^m$. Direct calculation shows that

$$Z_p(t;s) = 1 + \frac{((p^{-s}-1)(p^{-2}|V_f(\mathbb{F}_p)| + (p-1)p^{-s-3} + (p-1)p^{-5s-6} - p^{-6s-6}))}{(1-p^{-s-1})(1-p^{-6s-5})}.$$

⌟

▸ EXERCISE 1. Do the $p = 3$ case.

$Z_p(t;s)$ is a rational function meromorphic continuation to $\mathbb{C}$. This give a better way to compute.

4. Tools for geometry

4.1. Milnor fibers. Let $f : X \to \mathbb{C}$ be a holomorphic function where X is a smooth complex manifold of dimension d. Then

THEOREM 4.1 (Milnor). *If* $0 < n << \varepsilon << 1$ *and* $x \in f^{-1}(0)$, *then*

$$\overline{f} : B(x,\varepsilon) \cap f^{-1}(D(0,n) \setminus \{0\}) \to D(0,n) \setminus \{0\}$$

is a fibration.

DEFINITION 7. The data: (1) Fiber F_x, (2) Monodromy M_x with action of $H^i(F_x,\mathbb{Q})$ on it, it called the Milnor fiber.

BASIC MONODROMY CONJECTURE. If α is a pole of the zeta function there is $x \in f^{-1}(0)$ such that $e^{2\pi i\alpha}$ is an eigenvalue of M_x.

4.2. Relative motives with monodromy. Recall $K_0(Var_S)$ is the Grothendieck group of S-varieties

$$K_0(Var_S) = \mathbb{Z}\{[X \to S]\}/\sim$$

with formal addition and fiber product over S as multiplication and $\sim$ is the scissor relation.

REMARK. For any $x \in S$ there is a fiber map $K_0(Var_S) \to K_0(Var_{\mathbb{C}})$ via $[Y \xrightarrow{\pi} S] \mapsto [\mathrm{Fiber}_x(\pi)]$.

Let μ_d be the group of d-th roots of unity. From homomorphisms $\mu_{ab} \to \mu_b$ via $s \mapsto s^b$ we get a (pro-)group

$$\widehat{\mu} = \varprojlim \mu_d.$$

We say $X \to S$ has good $\widehat{\mu}$ if it is induced by a good μ_d action. And good μ_d action means that

(1) it is trivial on S

(2) each orbit is contained in affine $U \subset X$.

DEFINITION 8.

$$K_0^{\widehat{\mu}}(Var_S) = \mathbb{Z}\{[V \to S, \widehat{\mu} - \text{action}]\}/\simeq$$

where $\simeq$ is the scissor relation and an extra relation that for any d-dimensional representation of $\widehat{\mu}$:

$$[S \times V, \widehat{\mu}] \simeq [S \times \mathbb{A}^d, \text{trivial}].$$

Notation: $\mathbb{L} = [A^d \times S \to S, \text{trivial action}], M_S^{\widehat{\mu}} = K_0^{\widehat{\mu}}(Var_S)[\mathbb{L}^{-1}]$.

4.3. Arc spaces. [**5**] Recall the arc spaces by,

$$\mathcal{L}_n(X) = X(k[\varepsilon]/\varepsilon^{n+1}) = \mathrm{Mor}_{\mathcal{Sch}}(\mathcal{S}pec\, k[\varepsilon]/\varepsilon^{n+1}, X).$$

Given $f : X \to \mathbb{A}^1$ where X is smooth and $X_0 = f^{-1}(0)$, define

$$f : \mathcal{L}_n(X) \to \mathcal{L}_n(\mathbb{A}^1)$$
$$\varphi \mapsto f \circ \varphi = a_0 + \cdots + a_n\varepsilon^n.$$

The question is how do the fibers look like?

Let $\mathfrak{X}_n = \{\varphi \in \mathcal{L}_n(X) : f(\varphi) = a_n\varepsilon^n, a_n \neq 0\}$. Since $a_0 = 0$, $\varphi \in \mathfrak{X}_n$ maps into X_0 (so $\mathfrak{X}_n$ is relative to X_0).

We have a morphism $\pi : \mathfrak{X}_n \to \mathbb{G}_m$ via $\varphi \mapsto a_n$ and an action of $\mathfrak{X}_n$ on $\mathbb{G}_m$ via

$$\lambda.\varphi(\varepsilon) = \varphi(\lambda\varepsilon) = \lambda^n a_n \varepsilon^n.$$

The fiber of π over 1,$\mathfrak{X}_{n,1}$ has a good μ_n action.

Definition 9 (Motivic Igasa zeta function).

$$Z^{naive}(f;\tau) = \sum_{n\geq 1}[\mathfrak{X}_n \to X_0]\mathbb{L}^{-nd}T^n$$
$$Z(f,T) = \sum_{n\geq 1}[\mathfrak{X}_{n,1} \to X_0, \widehat{\mu}]\mathbb{L}^{-nd}T^n.$$

Remark.

(1) $Z^{naive}(f : \mathbb{L}^{-s}) = \int_{\mathcal{L}(X)} \mathbb{L}^{-\mathrm{ord})_t f(\varphi)} d\mu$.

(2) Motivic fiber $\psi_f := -\lim_{T\to\infty} Z(T)$.

5. p-adic and motivic coincide

Let $h : Y \to X$ be a log resolution of f. This means that h is proper, $h^{-1}(X_0)$ is a normal crossings divisor, $h : Y \smallsetminus h^{-1}(X_0) \xrightarrow{\sim} X \smallsetminus X_0$ is an isomorphism and that Y is smooth.

So let $\{E_i\}$ be the components of $h^{-1}(X_0)$. $E_I = \cap_{i\in I} E_i$, $E_I^0 = E_I \smallsetminus \cup_{i\notin I} E_j$. The are N_i and v_i's such that

$$h^{-1}f^{-1}(0) = \sum N_i E_i, h^*(K_X) = K_Y + \sum (v_i - 1)E_i.$$

Nearby fiber of $f \circ h$ is simpler: let $M_I = \gcd(N_i)_{i\in I}$. Locally around E_I^0, $f \circ h$ is given by $f \circ h = uv^{M_U}$. There is a μ_{M_I} Galois cover $\widetilde{E}_I^0 \xrightarrow{M_I:1} E_I^0$ given locally by

$$\{(z,y) \in \mathbb{A}^1 \times (E_I^0 \cap U) : Z^{M_I} = u^{-1}(y)\}.$$

Theorem 5.1. $Z(f : T) = \sum_{I\neq\varnothing}(\mathbb{L}-1)^{|I|-1}[\widetilde{E}_I^0 : \widehat{\mu}] \prod_{i\in I} \frac{\mathbb{L}^{-v_i}T^{N_i}}{1-\mathbb{L}^{-v_i}T^{N_i}}$.

THEOREM 5.2 (Denef). *For all good p,* $(p >> 0)$

$$Z_p(f;s) = p^{-d} \sum_{\varnothing \neq I} |\widetilde{E}_I^0(\mathbb{F}_p)| \prod_{i \in I} \frac{(p-1)p^{-sN_i - v_i}}{1 - p^{-sN_i - v_i}}.$$

COROLLARY 1. *For all* $p >> 0$ *set* $\mathbb{L} = p$ *then*

$$Z^{naive}(f;\mathbb{L}^{-s}) = Z_p(f;s).$$

Example 5.1. Consider the log resolution of $f = y^2 - x^3$. We need three blow-ups and three divisors E_1, E_2 and E_3 for which we can compute

	E_1	E_2	E_3	C
N_i	6	2	3	1
v_i	5	2	3	1

One can see that $\widetilde{E}_1^0$ is an elliptic curve with six points removes, having deck transformations $\mathbb{Z}_6$ and $\mathrm{Aut}(E) = E \times \mathbb{Z}_6$ where the only elliptic curve E can be is $E \cong \mathbb{C}/\mathbb{Z} \oplus \zeta\mathbb{Z}$ where $\zeta^3 = 1$. One can use the DL formula to compute $Z(f,T)$ and $Z^{naive}(T)$. ⌟

6. Monodromy conjecture

$e^{2\pi i(-1)}$, $e^{2\pi i(-5/6)}$ are monodromy eigenvalues of $f = y^2 - x^3$.

CONJECTURE 1 (Igasa). If α is a pole of $Z^{naive}(\mathbb{L}^{-s})$ then $e^{2\pi i\alpha}$ is monodromy eigenvalue of the Milnor fiber.

CONJECTURE 2. If α is a pole of $Z^{naive}(\mathbb{L}^{-s})$ then α is a root of the Bernstien polynomial $b_f(s)$ and the multiplicity of α as a pole of Z is less than or equal to the multiplicity of α as a root of b_f.

REMARK. The first conjecture is proves in cases $f : X \to \mathbb{A}^1$ and $\dim X = 2$ or $\dim X = 3$ and f is ??. The second conjecture is proves for $f : X \to \mathbb{A}^1$ and $\dim X = 2$ and $f = 0$ is reduced.

CHAPTER 5

Motivic McKay Correspondence

Atsushi

1. Definitions

Let $\mathbb{L} = [\mathbb{A}^1] \in K_0(V)$ be the Lefschetz motive and $\mathbb{L}^{-1}$ the Tate motive in

$$R := \overline{K_0(V)}[\mathbb{L}^{-1}]$$

the $\mathbb{L}$-adic completion. Also set X to be a variety and D and effective divisor on X. Recall the jet spaces

$$J_\infty(X) = \varprojlim J_k X \xrightarrow{\pi_k} J_k X.$$

We define the mapping $F_D : J_\infty X \to \mathbb{Z}_{\geq 0}$ via $\gamma \mapsto \gamma.D$. Recall the integration defined as the homomorphism

$$h(X, D) = \int_{J_\infty X} \mathbb{L}^{-F_D} d\mu := \sum_{n \in \mathbb{Z}_{\geq 0}} \mu({F^{-1}}_D(h))\mathbb{L}^{-n} \in R$$

where μ is the function from cylinders to R via ${\pi^{-1}}_k(B_k) \mapsto [B_k]\mathbb{L}^{k(n)}$.

Let $\varphi : Y \to$ be a normal crossing resolution and $D = \sum_{i \in I} a_i D_i$ the *discrepancy divisor*. We fix the notation that for $J \subset I$, $D_J = \cap_{j \in J} D_j$ and $D_J^\circ = D_J \smallsetminus \cup_{j \notin J} D_J$.

Definition 10. The stringy motive of X is the invariant of X,

$$h_{st}(X) = \int_{J \infty Y} \mathbb{L}^{-F_D} d\mu.$$

Remark. Robert showed that (by the change of variable formula) the above is equal to

$$h_{st}(X) = \sum_{J \subset J} [D_J^\circ] \prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{a_j + 1} - 1}$$

Example 1.1. For $D = aE$ where $a \in ?$

$$h_{st}(X) = [Y \smallsetminus E] + [E] \frac{\mathbb{L} - 1}{\mathbb{L}^{a+1} - 1} = [Y \smallsetminus E] + \frac{[E]}{[\mathbb{P}^a]}$$

⌟

Example 1.2. $n = ab$ and $G = \mathbb{Z}/b$ acting on $\mathbb{C}^n$ by the diagonal action

$$\zeta.v = \begin{pmatrix} \zeta & & \\ & \ddots & \\ & & \zeta \end{pmatrix} v; \text{ for any } \zeta = \exp({}^{2\pi i}/_{b}).$$

Let $\varphi : Y \to X = \mathbb{C}^n/G$ be the minimal resolution of the quotient. Then

$$h_{st}(X) = [Y \smallsetminus \mathbb{P}^{n-1}] + \underbrace{\frac{[\mathbb{P}^{n-1}]}{[\mathbb{P}^b]}}_{=1+\mathbb{L}^1+\mathbb{L}^{2a}+\dots+\mathbb{L}^{a(b-1)}}$$

⌟

2. Orbifold Euler numbers

Let G be a finite group and M a topological space with an action of G on it. We know that the classical Euler number of M/G is given by

$$e(M/G) = \frac{1}{|G|} \sum_{g \in G} e(M^g)$$

where M/G is the topological quotient space.

Definition 11. The orbifold cohomology of M/G is defined via

$$H_{orb}(M,G) = \bigoplus_{[g] \in G} H^*(M^g/C(g))$$

where $C(g)$ is the centralizer of g.

Any loop γ in M/G lifts (not uniquely) to $\widetilde{\gamma}$, which might not be closed anymore. But there is $g \in G$ such that

$$\widetilde{\gamma}_1(0) = g.\widetilde{\gamma}_1(1).$$

We say two lifts are equivalent $\widetilde{\gamma}_1 \sim \widetilde{\gamma}_2$ iff $\widetilde{\gamma}_1 = h\widetilde{\gamma}_2$. This implies that

$$\widetilde{\gamma}_2(0)(hgh^{-1})\widetilde{\gamma}_2(1).$$

From string theory we have that G acts on $\coprod_{g \in G} M^g$ $(h.M^g = M^{hgh^{-1}})$, and the quotient stack is just

$$[(\coprod_{g \in G} M^g)/G] = \coprod_{[g] \in G} (M^g/C(g)).$$

Definition 12. The orbifold Euler number is defined by

$$e(M,G) = \sum_{[g]\in G} (M^g/C(g)) = \frac{1}{|G|} \sum_{gh=hg} e(M^{\langle gh \rangle})$$
$$= \sum_{[H]\subset G} e(X^H) \times |\{[h] \in H\}|.$$

Here $X = M/G$ and $X^H = \{x \in X : \mathrm{Stab}_y(G) = H, y \in \pi^{-1}(x)\}$.

Example 2.1. For the action of $\mathbb{Z}/n$ on $\mathbb{P}^1$, $e(\mathbb{P}^1, \mathbb{Z}/n) = e(\mathbb{P}^1) + 2(n-1) = 2n$. Note that $e(\mathbb{P}^1/\mathbb{Z}/n) = e(\mathbb{P}^1) = 2$. ⌟

3. McKay correspondence

For a subgroup $G \subset \mathrm{U}(n)$, with an action of G on $\mathbb{C}^n$ such that the fixed points form a construtable set,

$$e(\mathbb{C}^n, G) = \#\{\text{ conjugacy classes in } G\ \}.$$

For instance if $G \subset \mathrm{SU}(2)$ the crepant resolution $Y \to \mathbb{C}^2/G$ has some k copies of (-2)-curves as the fiber over $0 \in \mathbb{C}^2/G$. Then

$$e(Y) = \sum_{i=1}^{k} e(E_i) - \sum_{i=1}^{k-1} e(E_i \cap E_{i+1}) = 2k - (k-1) = k.$$

The classical McKay correspondence states that $e(Y) = k+1 = e(\mathbb{C}^2, G)$. However for $n \geq 3$ it is not alway the case that a crepant resolution exists.

Theorem 3.1. *For finite $G \subset \mathrm{SL}(n)$, $\#G = r$ and an action of G on $\mathbb{C}^n$*

$$h_{st}(X) = \sum_{[H]\subset G} [X^H] \sum_{[g]\in H} \mathbb{L}^{\mathrm{age}(g)}$$

where if $g \sim \mathrm{diag}(\exp(2\pi i/r\alpha_1), \cdots, \exp(2\pi i/r\alpha_n))$ we define $\mathrm{age}(g) = \frac{1}{r}\sum_{i=1}^{n} \alpha_i$.

Corollary 2 (The classical McKay). $e(h_{st}(X))|_{\mathbb{L}=1} = e(\widetilde{\mathbb{C}^n/G}) = e(\mathbb{C}^n, G)$.

Proof of the theorem. First step is stratifying $J_\infty X$ as

$$J_\infty X = \coprod_{[H]\subset G} \coprod_{[g]\in H} J_\infty^{H.g} X.$$

In fact for $y \in M$ such that $\mathrm{Stab}_y(G) = H$ and $\pi(y) = x$, any arc $\gamma \in J_\infty X$ at $x = \pi_0(\gamma)$ lifts as

$$\begin{array}{ccc} M & \overset{\tilde{\gamma}}{\dashleftarrow} & \mathcal{S}pec\,\mathbb{C}[\![s]\!] \\ \downarrow \pi & & \downarrow t=s^r \\ M/G & \underset{\gamma}{\longleftarrow} & \mathcal{S}pec\,\mathbb{C}[\![t]\!] \end{array}$$

Unless $\tilde{\gamma}$ falls entirely in the branch locus of π, there is $g \in H$ such that $\gamma(\zeta s) = g.\gamma(s)$. $\tilde{\gamma}$ is unique up to conjugation (?) and so is g. This yields the stratification above (except that we are leaving off a set of measure zero).

$$J_\infty^{H,g} X = \{\gamma \in J_\infty X : \exists y \in M, \pi_0(\gamma) = \pi(y0, \mathrm{Stab}_y G = H\}$$

Example 3.1. $G = \mathbb{Z}/3 = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}, \begin{pmatrix} \zeta^2 & 0 \\ 0 & \zeta^2 \end{pmatrix}\}$. $X = \mathbb{C}^2/G = \mathcal{S}pec\, \mathbb{C}[x^3, y^3, xy] = \mathcal{S}pec\, \mathbb{C}[x, y, z]/(xy - z^3)$ and the fiber over zero is two copies of $\mathbb{P}^1$. The arc

$$\gamma(t) = (X(t), Y(t), Z(t)) = (a_1 t + a_2 t^2 + \cdots, b_1 t + b_2 t^2 + \cdots, c_1 t + c_2 t^2 + \cdots) \in \pi_0^{-1}(0)$$

satisfies

$$X(t)Y(t) = Z^3(t)$$

hence $\pi_0^{-1}(0) = C_1 \cup C_2$ with $C_1 = \{\gamma : b_1 = 0\}$ and $C_2 = \{\gamma : a_1 = 0\}$. The mentioned lifts of $\gamma_1(t) = (t, t^2, t)$ in C_1 and $\gamma_2(t) = (t^2, t, t) \in C_2$ are $\tilde{\gamma}_1 = (s, s^2), \tilde{\gamma}_2 = (s^2, s)$. Then

$$\tilde{\gamma}_1(\zeta, s) = (\zeta, \zeta^2 s^2) = g_1(s, s^2), \text{ and } \tilde{\gamma}_2(\zeta, s) = g_2(s^2, s).$$

⌟

The second step of the proof is the integration. ?? □

CHAPTER 6

The *Universal* theory of motivic integration

Gulia Gordon

Today I'll give a talk about the rationality of another generating series that occurs in arithmetic (closely related to Igusa series that appeared in Andrew Morrison's first talk). In the process, however, I'll mostly talk about the role of logic in the "universal" theory of motivic integration, and will explain some background behind the example of computing a p-adic Igusa zeta function that appeared in Andrew's first talk.

My sources are, approximately: [**4**], [**3**], [**2**], and the notes by Yoav Yaffe and myself.[1]

Recall that for $f \in \mathbb{Z}[x_1, \cdots, x_d]$, and p prime $N_{p,n}(f)$ is the number of solutions to $f(\overline{x}) = 0$ mod p^n. This gave us Igusa series

$$\sum_{n\geq 0} N_{p,n} T^n = Q(T).$$

The motivic analogue is

$$Q^{mot}(T) = \sum_{n\geq 0} [J_n(X)] T^n \in (K_0(\widehat{Var_k})[\mathbb{L}^{-1}])[T]$$

where $X = \{f = 0\}$ over $\mathbb{C}$. We proved that this is a rational function in T with denominator $\prod_{i=1}^{k}(1 - \mathbb{L}^a T^b)$. Recall that we related this series with $T = \mathbb{L}^s$ for $s \in \mathbb{C}$ to a motivic integral on X. And given a resolution singularities $Y \to X$ we computed the motivic integral using change of variables formula and got an explicit formula on Y.

Question: How can we show rationality of the Igusa series on the arithmetic side? The first part of the quesiton is to substitute the resolution of singularities with some tool in the $\mathbb{Z}_p$ or $\mathbb{F}_p[\![[]]\!]t]$ case. The second part will be answering the question about the Serre series.

Similar question: Let $\overline{N}_{p,n}(f)$ be the number of solutions to $f(\overline{x}) = 0 \mod p^n$ that lift to actual p-adic solution and set

$$P(T) = \sum_{n\geq 0} \overline{N}_{p,n}(f) T^n.$$

[1] http://www.math.ubc.ca/~gor/motivic.html.

The motivic analogue would be

$$P^{mot}(T) = \sum_{n\geq 0}[\pi_n(J_\infty(X))]T^n.$$

Note that when X is smooth, π_n is surjective and we don't get anything new. And the question is rationality of the above two series. The former was tackled by Denef in 80's.

1. Quantifier elimination

THEOREM 1.1 (Ax). *The language of algebraically closed fields has quantifier elimination.*

Definable sets: subset of $\mathbb{A}^n$ over our field that is defined by a formula in the theory of algebraically closed fields. A morphism $f : Y \to X$ has the image $\{x \in X : \exists y \in Y : f(y) = x\}$. By quantifier elimination this is equivalent to some formula $\varphi(x)$ without quantifiers. This shows that the image is a constructible set which is basically Chevalley's theorem.

Example 1.1 (Presburger language for $\mathbb{Z}$)**.** The language containing $0, 1, +, \geq, =$ and $\cong_d$ for every $d \in \mathbb{N}$ has quantifier elimination by a result of Macyntire. Godel shows that if you add multiplication then there is no quantifier elimination (incompleteness theorem). The definable subsets of $\mathbb{Z}$ are then the finite unions of points and arithmetic progressions. A *definable function* is one for which its graph is a definable set. Definable functions in Presburger language are linear combinations of piecewise-linear functions and the periodic ones. ⌟

2. Rationality of $Q(T)$

Here are the steps we will follow:

(1) There is a language for valued fields. In particular we will have an order map ord and ?? component ac.

(2) This language will let you work on structures

$$(\text{valued field})^n \times (\text{residue field})^m \times \mathbb{Z}^r$$

where by $\mathbb{Z}$ we mean the Presburger language.

(3) We construct a class of functions on the above structure which is stable under integration.

(4) Theorem: Suppose $\{A_m\}$ is a collection of definable subsets of $\mathbb{Z}_p^d$ that depends on m definably, i.e.

$$\bigcup_m A_m \subset \mathbb{Z}_p^d \times \mathbb{Z}$$

is definable. Then $\int_{A_m} 1(dx)_p$ is a definable function of m.

(5) Easy fact: Say $I(m)$ is a Presburger definable function on $\mathbb{Z}$. Then $\sum_m I(m)T^m$ is a rational function of T. Now take $I(m) = \int_{A_m} 1|dx|_p$.

3. Constructible motivic functions

3.1. Examples of valued fields. (1) Let p be a fixed prime. The p-adic valuation of $\mathbb{Q}$ is given via

$$v_p(a/b) = n, \text{ when } a/b = p^n a'/b/, (a',p) = (b',p) = 1$$

and we complete $\mathbb{Q}$ with respect to the corresponding absolute value

$$|a/b| = p^{-n}.$$

So we can think of elements of $\mathbb{Q}_p$ as series $a_n p^n + a_{n-1}p^{n-1} + \cdots$. As a piece of notation we let

$$ord(x) = v_p(x) \in \mathbb{Z}, ord(0) = +\infty$$

and the first nonzero coefficient in the p-adic expansion is denote by

$$\overline{ac}(x) = a_n \in \mathbb{F}_p, \overline{ac}(0) = 0.$$

The residue field in this example is $\mathbb{F}_p$.

(2) $\mathbb{F}_p(tt)$ with residue field $\mathbb{F}_p$.

(3) $\mathbb{C}((t))$ with

$$ord(f) = n, \overline{ac}(f) = a_n \in \mathbb{C}$$

when $f = a_n t^n + \cdots$. The residue field in this case is $\mathbb{C}$.

If k is a valued field, and m_k is the maximal ideal of ring of integers

$$\mathcal{O}_k = \{x : ord(x) \geq 0\}$$

$\mathcal{O}_k/m_k = k$ is our residue field.

3.2. Denef-Pas language. There are three sorts of variables:

(1) $x_i, y_i, \cdots$: variables of the valued field sort;

(2) $\zeta_i, \eta_i, \cdots$: variable of the residue field sort;

(3) $n_i, m_i, \cdots$: running over $\mathbb{Z} \cup \{\infty\}$.

The first two are thought of as if in the language of rings, and the third one in Presburger's language. ord is a formula from (1) to (3) and $\overline{ac}$ is from (2) to (3).

3.3. Constructible sets.

Example 3.1. $ord(x-1) \geq 2 \wedge \overline{ac}(x-1) = 1$ defines the set

$$\{\ 1 + p^2 + \text{ higher order terms }\}.$$

⌟

Example 3.2. Say X is a subvariety of $\mathbb{A}^d_{\mathbb{C}}$ cut out by

$$f(\overline{x}) \in \mathbb{Z}[\overline{x}], \overline{x} = (x_1, \cdots, x_d).$$

Then $\pi^{-1}{}_{\infty}(X)$, the level-zero cylinder

$$\begin{array}{ccc} J_{\infty}(\mathbb{A}^f) & \supset & \pi^{-1}_{\infty}(X) \\ \big\downarrow & & \big\downarrow \\ \mathbb{A}^d_{\mathbb{C}} & \supset & X \end{array}$$

can be defined by the formula

$$\varphi(\overline{x}) = \exists \overline{y}, ord(\overline{y}) \geq 0 \wedge f(\overline{y}) = 0 \wedge ord(\overline{y} - \overline{x}) > 0 \wedge ord(x) \geq 0.$$

Note that $J_{\infty}(\mathbb{A}^d)(\mathbb{C}) = \mathbb{A}^d(\mathbb{C}[\![t]\!])$. In fact all constructible sets are definable in our language. ⌟

4. Integration

The idea is to use "motivic integration" to go from formulas in DP (which define subsets of

$$(\text{ valued field })^n \times \mathbb{Z}^d \times (\text{ residue field })^k$$

to formulas with only variables running over $\mathbb{Z}$ and the residue field. Note that each of our three examples above had their own motivic integrations. The first two had p-adic valuations, and the third one had motivic volume.

Example 4.1. If $\varphi(mx) = \{\overline{x} : ord(f(x)) \geq m\}$,

$$I(m) = vol(\{x : ord(f(x)) \geq m\}).$$

⌟

5. Cell decomposition

How motivic integration works? Consider the mapping

$$\begin{aligned} \lambda : \mathbb{Q}_p \smallsetminus \{0\} &\to \mathbb{Z} \times \mathbb{F}_p \\ x &\mapsto (ord(x), \overline{ac}(x)). \end{aligned}$$

Cell decomposition [Denef, Pas] says that given a definable subset of $\mathbb{Q}_p$ you can find finitely many parameters $(\overline{n}, \overline{\zeta})$ such that our set is a (countable) union of the *p-adic balls*, $\lambda^{-1}(\overline{n}, \overline{\zeta})$.

Now say A is a definable subset of (valued field)^n. f is definable function on A. Then a *cell* is a disjoint union of one-dimensional balls (in t) of the form $\lambda^{-1}(\overline{\zeta}, \overline{n})$.

THEOREM 5.1. *One can break A into a finite union of $C_i \times (cell_i)$ where C_i's are subset of $\mathbb{A}^n$. so that on each $C_i \times$ cell, $f(x_1, \cdots, x_n, t)$ is a pullback of a function of $(x_1, \cdots, x_n)$.*

Then the integration in t is easy.

CHAPTER 7

Singularities of pairs and motivic integration

Andrew Staal

Idea: Jet schemes/motivic integration become useful tools in higher dimensional birational geometry.

Mori's program: How to generalize to higher dimension?

Techniques: consider singularities (X, Y), initially Y is a $\mathbb{Q}$-divisor, i.e. a multiple of it is Cartier. More precisely, let X be normal and K_X the push forward of the canonical line bundle on the smooth locus. Let Y be such that $K_X + Y$ is $\mathbb{Q}$-cartier.[1]

1. Some definitions

Recall that a log resolution of (X, Y) is a proper, birational map $X' \xrightarrow{f} X$, such that $F = f^{-1}Y = \sum a_i D_i$, and

$$K_{X'/X} = \sum b_i D_i$$

and $D = \sum D_i$ is SNC.

Let $q \in \mathbb{Q}$ then (X, qY) is Kawamata Log Terminal (KLT) if

$$b_i - qa_i > -1, \forall i$$

and is terminal (T), if

$$b_i - qa_i > 0, \forall i.$$

(X, qY) is Log Canonical (LC) if

$$b_i - qa_i \geq -1, \forall i.$$

Let the log canonical threshold of (X, Y) be defined as

$$\operatorname{lct}(X, Y) = \sup\{q : (X, qY) \text{ is KLT}\} = \min_i \{\frac{b_i + 1}{a_i}\}.$$

and

$$\operatorname{lct}(X, qY) := \frac{1}{q} \operatorname{lct}(X, Y).$$

[1] Recently Mustata takes Y to be any closed subscheme.

2. Relation to Jets

Let X be a smooth variety over $\mathbb{C}$.

THEOREM 2.1 ([**6**]).

$$\mathrm{lct}(X,Y) = \dim X - \sup_m \{\frac{\dim J_m(Y)}{m+1}\}.$$

This is an easy consequence of the following,

THEOREM 2.2. *Let* $\dim X = n$*,*

(1) (X,qY) *is KLT if and only if* $\dim J_m(Y), (m+1)(n-q)$ *for all* m*,*

(2) (X,qY) *is LC if and only if* $\dim J_m(Y) \leq (m+1)(n-q)$ *for all* m*.*

SKETCH OF THE PROOF.

$$\int_{\mathrm{ord}^{-1}{}_Y(m+1)} \mathbb{L}^{q\mathrm{ord}_Y} d\mu_X = \int_{\mathrm{ord}^{-1}{}_F(m+1)} \mathbb{L}^{-\mathrm{ord}_{K_{X'/X}-qF}} d\mu_{X'}$$

by the birational transformation rule. Now

$$\mathrm{ord}^{-1}{}_Y(m+1) = \psi^{-1}{}_n(J_m(Y)) \smallsetminus \psi^{-1}{}_{m+1}(J_m(Y))$$

by definition, and by definition now the left hand integral is

$$(([J_m(Y)] - [J_{m+1}(Y)]\mathbb{L}^{-n})\mathbb{L}^{-nm})\mathbb{L}^{q(m+1)}.$$

□

3. Contact loci

Let $\gamma : \mathcal{S}pec\,\mathbb{C}[[t]] \to X$ be an element of the jet space. Then

$$\mathrm{ord}_Y(\gamma) = e \Leftrightarrow \mathcal{I}_Y.\mathbb{C}[[t]] = \langle t^e \rangle.$$

We define

$$\mathrm{Cont}^p(Y) = \{\gamma \in J_\infty(X) : \mathrm{ord}_Y \gamma = p\}$$

called the *contact loci* and if $p \leq m$,

$$\mathrm{Cont}^p(Y)_m = \{\gamma \in J_m(X) : \mathrm{ord}_Y \gamma = p\}.$$

Let $E = \sum E_i$ be a SNC divisor, and $\nu \in \mathbb{N}^k$ a multi-index, then

$$\mathrm{Cont}^\nu(E) = \{\gamma \in J_\infty(X) : \mathrm{ord}_{E_i}(\gamma) = \nu_i, \forall i\}$$

is the *multi-index* contact loci.

The use of these is in introducing a new birational-transformation rule as follows. Given a log resolution $X' \xrightarrow{f} X$, we have induced map

$$f_\infty : J_\infty(X') \to J_\infty(X)$$

and for all p,

$$\mathrm{Cont}^p(Y) = \coprod_{\nu} f_\infty(\mathrm{Cont}^\nu(E))$$

where ν ranges over all sums $\sum \nu_i p_i = p$. Moreover, each $f_\infty(\mathrm{Cont}^\nu(E))$ and a constructible cylinder of codimension $\sum \nu_i(b_i + 1)$.

For all irreducible components $Z \subseteq \mathrm{Cont}^p(Y)$, there is a unique ν for which $f_\infty(\mathrm{Cont}^\nu(E))$ is dense in Z. In the setup of motivic integration

THEOREM 3.1 (Denef-Loeser). *For all $e \in \mathbb{Z}_{\geq 0}$ and $m \geq 2e$, $\mathrm{Cont}^e(K_{X'/X})_m$ is a union of fibers isomorphism to $\mathbb{A}^e$ and if γ', γ'' are in some fiber then ??*

COROLLARY 3.

$$\mathrm{lct}(X, Y) = \dim X - \sup_m \left(\frac{\dim J_m(Y)}{m+1} \right).$$

COROLLARY 4. *If $Y \subset X$ is a reduced, irreducible local complete intersection subvariety, then the following are equivalent*

(1) $J_m(Y)$ is irreducible for all m,

(2) Y has rational singularities,

(3) Y has canonical singularities.

CHAPTER 8

Bernstein polynomials

Bill Casselman

This is a talk in analysis. The problem we want to tackle is this: given $p(x)$ a polynomial in $\mathbb{R}^n$, and $f \in C_c^\infty(\mathbb{R}^n)$, the question Gelfand asks in 1984 in ICM is whether

$$\int_{\{p(x)>0\}\subseteq\mathbb{R}^n} p(x)^s f(x)dx$$

continues meromorphically and do the poles have arithmetic progression.

Example 0.1. $p(x) = x$ and $n = 1$, then the integration of $\int_0^\infty x^s f(x)dx$ goes back to Euler and by integration by parts is equal to

$$\frac{1}{s+1}\int x^{s+1} f'(x)dx = \cdots = \frac{\pm 1}{(s+1)\cdots(s+n)}\int x^{s+n} f^{(n)}(x)dx = \cdots$$

so it continues to $\mathbb{C}$ and has simple poles at $-\mathbb{N}$. This generalizes to the Gamma function,

$$\int_0^\infty x^s e^{-x} dx = \Gamma(s).$$

⌟

A slightly more interesting example is due to Riesz:

Example 0.2. Let $p(x) = x_1^2 \cdots x_n^2$. Then

$$\int p(x)^s f(x)dx = \frac{1}{4(s+1)(s+\frac{n}{2})}\int p^{s+1} f(x)dx.$$

What you actually need to solve is the differential equation

$$\Delta r^{2s} = 4s(s+1-n/2)r^{2s}.$$

⌟

This is almost as much as classical examples can go. Singularities make the problem hard. There are two papers of Atiyah and Bernstein [1969-79] use resolution of singularities to prove Gelfand's conjecture. There is a correspond with distributions as is apparent from the last example:

$$\langle \Phi', f\rangle = -\langle \Phi, f'\rangle.$$

All goes through if we can find solution to

$$D = \sum_{\alpha \in \mathbb{N}^n} C_\alpha(s,x) \frac{\partial^\alpha}{\partial x^\alpha}$$
$$DP^s = B(s)P^{s-1}$$

THEOREM 0.2 (Bernstein). *For every P, D there is a polynomial $B(s)$,*

$$DP^s = B(s)P^{s-1}.$$

Let F be a field of characteristic zero, We have a slightly non-commutative Weyl algebra generated by x_i's and $\frac{\partial}{\partial x_i}$ satisfying

$$[\frac{\partial}{\partial x_i}, x_i] = \delta_{i,j}.$$

And Bernsteins's theorem is all about the case $F = \mathbb{C}(s)$. We denote the one generated by m, x_i's and ∂_i's, $\mathfrak{a}_n^m$. We have a grading by the order of differentials and

$$\mathfrak{a}_n^k . \mathfrak{a}_n^\ell \subseteq \mathfrak{a}_n^{k+\ell}.$$

Notation: $F[x, \xi]$.

Observe that $\mathfrak{a}_n$ is hence noetherian.

Question: What do the modules over $\mathfrak{a}_n$ look like?

Example 0.3. (1) $F[x]$ is a simple one. (2) ?? (3) Suppoe P is a polynomial, $s \in F$, and $\mathfrak{a}_n$ acts on a module generated by P^s via

$$x_i.P^s = x_i P, \frac{\partial}{\partial x_i} P^s = sP' P^{s-1}.$$

For instance for $P(x) = 1$ we get back $F[x]$. ⌟

Example 0.4. For $n = 1$ and $P(x) = x^s$ and $s = 0$ we get $F[x]$ and if $s = -1$, we get a module that has submodule $F[x]$ and the quotient of it by $F[x]$ is called Dirac's module, which is generated by the principal parts $\sum_{k>0} a_k / x^k$. ⌟

With $s \notin \mathbb{Z}$, $M_{p,s}$ is irreducible module and we have

THEOREM 0.3 (Bernstein). *$M_{p,s}$ has finite length as an $\mathfrak{a}_n$-module.*

Start with any module M, and look at "good" filtrations $\mathfrak{a}^n M^\ell \subset M^{k+\ell}$ then M^k / M^{k-1} has finite dimension. So we can look at the Hilbert polynomial, $H(m)$ of $\dim M^m$'s for large enough $m >> 0$. We have

THEOREM 0.4 (Bernstein). *The characteristic variety of M corresponding (via support) to the annihilator of $G(M)$ in $F[x, \xi]$, had dimension $\geq n$.*

M is called holonomic if the characteristic variety has dimension n. (In some sense they are all of finite length $\leq e$.)

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