GEOMETRIC INVARIANT THEORY

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1. Introduction

Lecture 1

Although GIT may look like a fusty old topic at the time, it is charming, useful and still alive. As a matter of the fact that it is an old subject, there are many good books (Newstead, Dolgachev, Mukai, Mumford) that can serve as reference. The subject is nowadays very large and its perhaps best to think of it as a subset of the area of algebraic transformation groups. There are new tools and new applications (e.g. computational complexity).

Plan of the four talks is as follows:

(1) What is GIT and where did it come from?
(2) Why does GIT work and how is it typically used?
(3) Refinements of the theory, applications to moduli questions and interesting cohomological features and developements.
(4) Variations and applications.

2. Historical overview

As Mumford puts it in the beginning of his book

> The purpose of GIT is to study two related problems. When does an orbits pace of an algebraic scheme acted on by an algebraic group exit? And to construct moduli schemes of various types of algebraic objects. The second problems appears to be, in essence, a special and highly non-trivial case of the first problem.

In fact this statement is always birationally true, but biregularly it may work, and when it works it works excellently, but it might fail and then we need to do some more work.

Systematically we are beginning with some classification problem. And we hope to reformulate it to a problem about group actions. The with the group action we want to pass to taking quotients to get orbit spaces (parametrizing orbits). And we want to think of this

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outcome as the moduli, the answer to the classification problem we started with. There are a few remarks about this process however.

Remark. At each stage we tend to lose some information. This issue is largely remedied by stacks.

Remark. The procedure of taking quotient is "topological".

(1) There are non-Hausdorff behaviors;

(2) and we lose information about the action (e.g. The orbit space of $\mathbb{C}^n$ by the action of S_n is again just $\mathbb{C}^n$);

(3) How to stay in category when taking the quotient?

Example 2.1 (In linear algebra)**.** Say we want to classify the linear transformations up to similarity. Pick a basis, let $A \in M_{n\times n}$ be the matrix of given linear transformation. Then $A \sim BAB^{-1}$ for any $B \in \mathrm{Gl}(n)$. Let $n = 2$ and work over $\mathbb{C}$. Then $M_{2\times 2}$ decomposes into union of Gl(2)-orbits. The question is how to parametrize orbits? Answer: try to separate orbits with invariant functions; Given 2 orbits find G-invariant f such that $f(O_1) = 0$ and $f(O_2) = 1$. By continuity the closed orbits are crucial. This case turns out to be pretty easy. det and tr are invariant functions. So we have a well-defined function

$$\varphi = (\det, \mathrm{tr}) : M_{2\times 2} \to k^2$$

and we ask what are the fibers. The general fiber is when we have distinct eigenvalues λ_1 and λ_2. Here we are good, getting 2-dimension closed orbit. In the special case of $\lambda_1 = \lambda_2 = \lambda$. The canonical form of such matrix is $\begin{pmatrix} \lambda & t \\ 0 & \lambda \end{pmatrix} (t \neq 0)$ but these are all similar to $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ so we have a 2-dimensional orbit which is not closed and another orbit in this fiber is $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ which is closed and in the closure are the first type orbit. ⌟

▸ **Exercise 1.** φ is surjective, but show that det and tr generate the ring of invariants.

The key observations to learn from the example above are as follows:

Remark. (1) Fiber over 0 (you may call it a nonlinear kernel) is distinguished in the sense that it is preserved under rescaling of the coordinates. So we call it the nullcone.

(2) The orbit space has double line but the "quotient" via invariant functions is $\mathbb{C}^2$.

(3) We have used elementary symmetric polynomials rather than the eigenvalues.

(4) If $n > 2$ we have similar situation: in the special fiber we have finitely many orbits but only a unique closed orbit in each fiber, i.e. the invariant functions separate the closed orbits.

(5) Over $\mathbb{Q}$ more refined orbit poset but still one closed orbit (so we have an independence of arithmetic, so we can have good behavior under field extensions and we can get topological properties such as fundamental groups out of this).

(6) det and tr form basis of invariants with special geometric properties.

Example 2.2 (In Euclidean geometry)**.** In $\mathbb{R}^2$ classify conics. Euclidean motions $O(2) \ltimes \mathbb{R}^2$. Here if Δ is the discriminant, then the two functions that assign to a conic Γ given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

the two numbers

$$f_1 = (a+c)\Delta^{-1/3}, f_2 = (ac - b^2)\Delta^{-2/3}$$

distinguish all the closed orbits. ⌟

Example 2.3 (in projective geometry)**.** Moduli of hypersurface from $\mathbb{P}\mathrm{GL}_n$ action on $\mathbb{P}^N$. The ring of invariants again should detect properties independent of the coordinates. So they are geometric properties by the 19-th century philosophy of isometris.

▸ **Exercise 2.** For smooth, this agrees with abstract isomorphism of varieties.

We can go further and work with configurations of hypersurfaces, for instance four points in $\mathbb{P}^1$ and $\mathbb{P}$GL-action on $(\mathbb{P}^1)^4$. Invariants give geometric properties of configuration of hypersurfaces. In principle relations among invariants gives you all theorems in projective geometry, in principles! ⌟

In general if we work with subgroups, we get other geometries: euclidean, affine, etc. (Klein's Erlangen program: algebraic geometry of nineteenth century is invariant theory). This was the setting for Hilbert. He was studying special classes of representations of Sl_n. Here is what he did. The broad idea of all the problems as we said is to use a complete set of invariants (continuous functions, thus picking out closed orbits) to define quotient $\Rightarrow$ categorical quotient, having universal mapping property.

But we have to be very careful even in algebraic geometry with our choice of category. Here are two examples of what can happen

Example 2.4. $\mathbb{G}_a = \{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \neq 0\} \subset \mathrm{Sl}_2$ acts on Sl_2 via

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+tx & b+td \\ c & d \end{pmatrix}$$

so c and d are the invariants. What does the invariant map give us? It is a map $\varphi : \mathrm{Sl}_2 \to \mathbb{A}^2$ just like the GL_2 action example above, but now we don't hit $(c,d) = (0,0)$. So the surjection is to $\mathbb{A}^2 \smallsetminus \{0\}$ which is quasi-affine but not affine. This is very different from what happened in previous example. ⌟

▸ **Exercise 3.** Take a smooth affine quadric surface Q_2. Show that there is a $\mathbb{G}_a$-action such that Q_2 maps surjectively on the double line. Observe that the action is set theoretically free.

Recap of the above idea: list invariants $f_1, \cdots, f_s$. Define $\varphi = (f_1, \cdots, f_s) : X \to \mathbb{A}^s_k$. Take $\overline{\operatorname{im}\varphi}$. This is an affine variety with coordinate ring $k[f_1, \cdots, f_s]$. So $\overline{\operatorname{im}\varphi}$ "should" be the categorical quotient in the category of affine varieties. But the problem is that we can only make this canonical if the ring of invariants is finitely generated. This was the big issue in Hilbert's days. In 1890 Hilbert proved finite generation for his class of actions via Hilbert basis theorem. More generally the Hilbert's 14th problem asks whether there exists a linear representation of linear algebraic group G such that ring of invariants in not finitely generated. Others (in particular Nagata) applied Hilbert's technique to a large class of groups called the reductive groups (i.e. maximally connected solvable normal subgroups of tori; so really we are talking about things like finite groups, semi-simple groups and tori themselves).

Theorem 2.1 (Nagata)**.** *There is a counterexample to Hilbert's 14th.*

Theorem 2.2 (Popov)**.** *If G not reductive, then you can always find a counterexample for that G.*

Zariski's nice generalized version of Hilbert 14th: If R is an affine ring over k, L' is an intermediate field $k \subseteq L' \subseteq L = k(R)$, where $k(R)$ is the field of fractions of R, when is $R \cap L'$ an affine ring? An example of Zariski and Mumford: D divisor of X, and X is projective. $R(X, D) = \oplus_{m \geq 0} H^0(X, L_D^{\otimes m})$. When is this finitely generated?

Theorem 2.3 (Winkelmann, 2004)**.** *This is equivalent to the original 14th problem for non-reductive groups in the sense that you can find X and G such that $k[X]^G$ is not finitely-generated.*

Hilbert's argument was criticized by his referee for the paper, Gordan:

> "This is not mathematics, this is theology".

Klein on the other hand guaranteed that this paper will be published without any changes. Hilbert worked hard three more years to make the argument constructive resulting in Hilbert's paper of 1893 which is one of the core papers in the history of algebraic geometry. The crude idea is that we have the generators $f_1, \cdots, f_s$. And consider the variety associated to the ideal $(f_1, \cdots, f_s)$ or the nullcone. His naive hope was to find another way to describe nullcone, and work backwards to complete the basis $f_1, \cdots, f_s$. This does not work since the geometric object, the nullcone, corresponds not to the ideal but to the radical (⇒Nullstellensatz). But it is sufficient to get a system of parameters (⇒Noether normalization). But we need a bound for the degree of the map, and there he failed.

His idea of looking at the nonlinear kernel as a variety is the beginning of the geometric invariant theory and to be honest, it is the geometric invariant theory.

Later on many people worked on the degree bound of Hilbert's question. Noether did it for finite groups. Popov did the case of semi-simple groups in 1981. And the case of tori was done by Kempf in 80s and Wehler in 1983. As a result there is an algorithm for computing the generators of invariants of reductive group actions if we put these results together.

Theorem 2.4. *Let k be a field of characteristic zero and G reductive. Then there is an algorithm to compute the invariants in the spirit of Hilbert.*

Later, after the usefulness of Hilbert's method was universally recognized, Gordan himself would say

> "I have convinced myself that even theology has its merits."

3. GIT in action

Lecture 2

Recall our prototype moduli problem of classifying linear endomorphisms with the following crucial features:

(1) Map of invariants: $\varphi : M_{n\times n} \to \mathbb{A}^n_k$ given by elementary symmetric polynomials in eigenvalues;
(2) $k[M_{n\times b}]^{\mathrm{Gl}_n} \cong k[\mathbb{A}^n_k]$.
(3) φ is a categorical quotient (in the category of affine varieties) written as $M_{n\times n}/\!\!/\,\mathrm{GL}_n \cong \mathbb{A}^n_k$.
(4) φ separates closed orbits: there is exactly one closed orbits in each fiber in other words given 2 closed orbits there is an invariant function that separates them.
(5) Extending base field doesn't changes essentials.
(6) $\varphi^{-1}(0)$ is the *nullcone* which in particular is a cone and consists of a closed orbit which is a point, together with the orbit of say $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in the $n = 2$ case. Observe that every point in this orbit is sent to the closed point by a one-parameter subgroup.

This example seems quite special; what about other groups?

Theorem 3.1 (Chevalley)**.** *If k is a perfect field and G an algebraic group, there is an abelian variety (in particular projective) A and a linear algebraic group (in particular affine) H fitting into the short exact sequence*

$$1 \to H \to G \to A \to 1.$$

And furthermore we have the decomposition $H \cong H_r \ltimes H_u$ to a reductive (e.g. finite, semi-simple, tori) H_r and unipotent H_u (e.g. extension by $\mathbb{G}_a$'s, ...). The latter is isomorphic to an affine space $\mathbb{A}^n$ as a variety, and the orbits of an action of the unipotent group on an affine variety are closed.

4. Group actions

It is convenient to express the group actions in terms of graphs: $\psi : G \times X \to X \times X$ via $(g,x) \mapsto (x, g.x)$.

Definition 1. The action is

(1) separated if $\mathrm{im}(\psi)$ is closed;

(2) proper if ψ is proper;

(3) free (in the scheme theoretic sense) if ψ is a closed immersion.

Remark. (2) does not mean that the quotient is proper! (Consider $\mathbb{C}^*$ acted on by $\mathbb{C}^*$ for instance.)

Remark. Free actions give principal G-bundles but in general the base need not be a scheme. But if G is reduced and acts on affine X, then we get a scheme as base.

Recall the counterexamples to the prototype behaviors:

(1) $\varphi : X \to \mathbb{A}^s_\ell$ need not send G-invariant closed subsets to closed subsets; e.g. $\mathbb{G}_a$ action on $\mathrm{SL}_2 \to \mathbb{A}^2$ gives qausi-affine and not affine subscheme $\mathbb{A}^2 \smallsetminus \{0\}$.

(2) $\varphi^{-1}(pt)$ need not contain a closed orbit, i.e. invariant functions do not necessarily separated closed orbits, e.g. consider Sym^2 of the defining representations for Sl_2, $\mathbb{G}_a \hookrightarrow \mathrm{Sl}_2$ acting on $\mathbb{C}^3$. The generic orbits are parabolas in horizontal planes $z = c \neq 0$. Orbits in $z = 0$ are parallel lines. In the 3-space if we consider a smooth affine quadric $y^2 - 2xz = 1$ is a $\mathbb{G}_a$-invariant hypersurface Q_2. The action of $\mathbb{G}_a$ on Q_2 has an orbit space of line with double point in the origin. Observe further that the action is set theoretically free, not separated, and has categorical quotient $\mathbb{A}^1$.

Application to naive Zariski cancellation: Does $X \times \mathbb{A}^1 \cong Y \times \mathbb{A}^1$ imply $X \cong Y$? The answer is often yes, but the equation is not so easy to solve for "nearly rational" varieties. Example: Q is the result of gluing two copies of $\mathbb{A}^1 \times \mathbb{A}^1$ along one of the $\mathbb{A}^1$'s. If C_2 is now a cubic hypersurface, from the fiber product

$$\begin{tikzcd} Q_2 \times C_2 \arrow[r] \arrow[d] & C_2 \arrow[d] \\ Q_2 \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

and the fact that there is exactly one $\mathbb{G}_a$-principal bundle on an affine variety, we should have

$$Q_2 \times \mathbb{A}^1 \cong Q_2 \times C_2 \cong C_2 \times \mathbb{A}^1.$$

But Q_2 and C_2 are not even homeomorphic (check that π_1^∞'s are different).

There is another unique example in Mumford's book in chapter one that shouldn't be considered as a counterexample of the prototypes but was contradicting many of the ideas that Mumford had hoped for.

> "The example shattered over-optimistic conjectures the authors entertained."

He considers an Sl_2-action on a smooth quasi-affine 4-dimensional variety. He shows that the action is set theoretically free; separated; geometric quotient exists (and indeed is the affine line) and yet the action is not proper. The variety is an affine surface in Sym^4 with $\mathbb{G}_a$-action.

5. Good quotients

To get a "good" quotient you want invariant functions to separate closed orbits. The simplest thing to do is to constrain yourself to groups that always do this! And this basic insight results in all the classical GIT.

Definition 2. A group G is geometrically reductive if for any affine G-variety, given any two closed disjoint G-invariant subsets S_1 and S_2, there is an invariant f such that $f(S_1) = 0$ and $f(S_2) = 1$.

The basic theorem is now that,

Theorem 5.1 (Haboush and others)**.** *G is geometrically reductive iff reductive (for any field) iff linearly reductive (i.e. linear representations are completely reducible) (in characteristic zero). (Nagata: in characteristic p all linearly reductive groups are finite group extensions of tori.)*

This gives us functoriality results with respect to closed immersions:

Theorem 5.2. *If* $\operatorname{char} k = 0$ *(this is not needed), G reductive, X an affine G-variety and Y a closed G-subvariety, then*

$$k[Y]^G = (k[X]/I(Y))^G = k[X]^G/I(Y) \cap k[X]^G,$$

i.e. every G-invariant function on Y is the restriction of a G-invariant function on X.

This fact is what makes GIT simple to use! Invariant functions are picking out closed orbits. We might call closed orbits *stable*. And thus we are getting functoriality of stability: closed in Y iff closed in X; i.e. stable in Y iff stable in X.

Remark. False for example in case of open inclusions, e.g. $\mathbb{C}^*$ acting on $\mathbb{C}$.

However there is a very different character to these closed/stable orbits. For example they might be of different dimensions. This implies for instance that can get bad singularities in the quotient if dimensions differ largely. Furthermore if dimension of the orbit is equal to the dimension of G (maximal dimension), then the action of G on that orbit is proper and one can show that (a) the space of such orbits forms a Zariski open in X such that (b) the action is proper on this subset. This is what nowadays is called the *stable* set but Mumford called it *properly stable*. The other orbits of an affine X are called semi-stable, $X = X^{ss}$.

By functoriality $Y^s = X^s \cap Y$. So somehow we can reduce to ambient affines. The first question you have to ask is now when do we get these nice G-equivariant embedding? The answer is always! And once we have this, we are reduced to studying linear actions on the projective space.

Theorem 5.3. *If G is an algebraic group acting on affine X. Then there exists a linear representation $T : G \to Gl(V)$ and a G-equivariant closed immersion $i : X \hookrightarrow V$.*

Theorem 5.4 (Kambayashi)**.** *If X is quasi-projective and normal, there exists a G-equivariant embedding $i : X \hookrightarrow \mathbb{P}(V)$.*

▸ **Exercise 4.** You do need normality; nodal cubics with $\mathbb{G}_m$ action cannot be linearized.

Significance of the null-cone: if X is a projective variety, with a linear G-action on the affine cone $\widehat{X}$, origin is cone point. We get invariant generators that are homogeneous and are vanishing at origin, so we define our map

$$\varphi = [f_1 : \cdots : f_s]$$

that is not defined where all the generators vanish. We get that the invariant function separate closed orbits. There is a unique closed orbit, namely the origin in the null-cone hence some point $\widehat{x}$ is in the null-cone iff $\overline{G.\widehat{x}} \cap \{0\} \neq \varnothing$ and φ is a rational map not defined precisely on the null-cone.

So we have three loci: $\widehat{X}^s \subseteq \widehat{X}^{ss} \subset \widehat{X}$, and $\widehat{X}^{uns} = \widehat{X} \smallsetminus \widehat{X}^{ss}$ which is just the null-cone and we will be throwing it out when taking quotient.

Theorem 5.5 (Mumford)**.** *If Y is a normal projective G-variety. Then there is a G-equivariant projective linear embedding $Y \hookrightarrow \mathbb{P}^n$ with the action of G on $\mathbb{P}^n$ such that $Y^s = Y \cap (\mathbb{P}^n)^s$ (likewise with semistables). Y^s/G is a variety (the geometric quotient) and $Y^{ss}/\!\!/G$ is a projective variety (the categorical quotient) and it suffices to pass to algebraic closure, so look at geometric points.*

So we shall study $v \in V$ such that $\overline{G.v} \cap \{0\} \neq \varnothing$. And this is the case iff there is a 1-parameter subgroup $\mathbb{G}_m \subset G$ such that $\overline{\mathbb{G}_m.v} \cap \{0\} \neq \varnothing$.

Twin aspects of GIT: (1) We have a choice of stability (based on choice of the embedding in the projective space), and that (2) given that choice, the nullcone is determined by $\mathbb{G}_m$'s in G. So morally GIT works well because it is the theory of multiplicative groups, covering spaces in a sense (tori and Weyl groups).

Recall our setting by the following example. We had two cases: Lecture 3

5.1. **The affine case.**

Example 5.1. X affine G-variety ($\operatorname{char} k = 0$, G reductive). Say $X = \mathbb{A}^2$ and $G = \mathbb{G}_m$ the action given by the representation $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. We have 3 types of orbits:

- closed of maximal dimension (called stable),
- closed of lower dimensions (strictly semistable),
- non-closed (strictly semistable).

Recall that the nullcone is the x and y axis in this example. There is a surjective morphism φ to $\mathbb{A}^1$ constructed from invariant functions. Recall that $X \xrightarrow{\varphi} \mathbb{A}^1$ is a categorical quotient. Recall that the closed orbits (the first two items above) are parametrized by the categorical quotient. We are allowed to think of all points (including the nonclosed orbits of the null-cone) as semistables because there is a non-vanishing invariant function at every point and X is an affine variety. ⌟

More generally,

$$X^s = \{x \in X : G.x \text{ closed in } X \text{ with 0-dimensional isotropy group }\}.$$

Fact: X^s is Zariski open subset of X on which G acts properly.

Remark. This Zariski open may be empty. An example is our prototype $M_{2\times 2}$ with the GL(2)-action. In fact both the group and the variety are four dimensional but we do not have any four dimensional orbits.

So the general picture is

$$\begin{tikzcd} X^s \arrow[d] \arrow[r, phantom, "\subseteq"] & X \arrow[d, two heads] \arrow[dr, "\varphi"] & \\ X^s/G \arrow[r, hook, "\text{open}"'] & X /\!\!/ G \arrow[r, hook, "\text{closed}"'] & \mathbb{A}^s \end{tikzcd}$$

X^s/G is the orbit space and is already a variety.

We saw that we can always reduce to linear representations by two reasons: equivariant embedding theorem and functoriality of stability under closed immersions.

$$G \circlearrowright X \xhookrightarrow[\text{equiv.}]{} \mathbb{A}^n \circlearrowleft \mathrm{GL}_n$$

Here $V = \mathbb{A}^n$ is our vector space and moreover $X^s = V^s \cap X$ (and likewise for semistables).

5.2. **The projective case.** If X is a projective normal (often not needed) G-variety over a field of characteristic zero and again G is reductive but acting projective linearly on an ambient projective space (normality implies this can be arranged). The lift of X to $\widehat{X}$, the affine cone, has the origin as a closed orbit and also the cone point. We also have a lift of $\varphi : X \to \mathbb{P}^s$ to $\widehat{\varphi} : \widehat{X} \dashrightarrow \mathbb{P}^s$ where $\widehat{\varphi} = [f_0 : \cdots : f_x]$ is a rational map:

$$\begin{tikzcd} \widehat{X} \arrow[d] \arrow[dr, dashed, "\widehat{\varphi}"] & \\ X \arrow[r, dashed, "\varphi"] & \mathbb{P}^s \end{tikzcd}$$

Key idea: $\widehat{\varphi}$ is a morphism precisely on the complement of the null-cone, which we now use a new name for it and call it the *unstable* locus. Here is again our big picture

$$\begin{tikzcd} X^s \arrow[d] & \subseteq & X^{ss} \arrow[d] & \subset & X \arrow[dr, "\varphi"] & \\ X^s/G \arrow[rr, hook, "open"'] & & X^{ss}/\!\!/G \arrow[rrr, hook, "closed"'] & & & \mathbb{P}^s \end{tikzcd}$$

Strictly speaking by definition the unstable locus, X^{uns} is the closed subvariety of X on which all homogeneous invariants vanish. Note that X^s/G is the orbit space and a variety and that the categorical quotient $X^{ss}/\!\!/G$ is projective. In many cases people abuse the notation and denote this categorical quotient by $X/\!\!/G$ which is often misleading.

Now we need a way to describe the null-cone.

Lemma 1. *$\widehat{x}$ is in the null-cone iff $0 \in \overline{G.\widehat{x}}$.*

Proof. If 0 in not contained in $\overline{G.\widehat{x}}$, since G is reductive and $\widehat{X}$ is affine there is a homogeneous invariant that is 1 on $\widehat{x}$ and zero on the origin. For the other way note that homogeneous invariants vanish on the origin and functions are constant on $G.\widehat{x}$ so they vanish on $\overline{G.\widehat{x}}$ by continuity. □

Functoriality of stability ($X^s = X \cap (\mathbb{P}^n)^s$ likewise for semistables and unstables) reduces our problem to study of the null-cone of a projective linear representation. The above lemma tells us that this is a union of orbits with zero in their closure.

Theorem 5.6. *If G is reductive, acting linearly on V, and v in the null-cone. Then there is $\mathbb{G}_m$ in G such that $0 \in \overline{\mathbb{G}_m v}$.*

This implies the Hilbert-Mumford numerical criterion: One can build the null-cone by G-sweep of the unstable set for tori in G. For instance if rank of G is one (hence the maximal torus is $\mathbb{G}_m$) the action of torus is always by

$$\begin{pmatrix} \lambda^{a_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{a_n} \end{pmatrix}_{n \times n}$$

in appropriate basis. Hence if all $a_i > 0$ or all $a_i < 0$ everything is going to be unstable. In fact we can characterize the unstable locus by weights of the linear representation. Same analysis works for semistable locus (with non-strict inequalities this time).

Example 5.2. Consider the action of the 3-dimensional representation of $\mathbb{G}_m$ via

$$\lambda \mapsto \begin{pmatrix} \lambda & & \\ & \lambda^{-1} & \\ & & 1 \end{pmatrix}$$

on $\mathbb{A}^3$. Show that the quotient is a $\mathbb{P}^2$, the x and y axis are unstable, the parallels to them in each $z = c$ plane is semistable and the rest of the orbits are stable. ⌟

Remark. The null-cone is the "worst" fiber in a number of ways, for instance it must be maximal dimensional. There is a theory of asymptotic cones dues to Borho and Kraft that associates to any fiber a cone that lives in the null-cone. As a consequence if the null-cone has finitely many orbits then all of the fibers have finitely many orbits.

The point that has to stressed is that not all $\mathbb{G}_m$'s in G are created equally. Some steer $\widehat{x}$ in the null-cone to 0 the fastest that should be distinguished in some way. Consider the action on $\mathbb{P}^1$ that comes from the linear action of $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ on the 2-plane. This has two unstable points corresponding to the two coordinate axis. Now look at the projective action of $\mathbb{G}_m$ via the representation

$$\begin{pmatrix} \lambda^2 & & \\ & \lambda & \\ & & \lambda^{-1} \end{pmatrix}$$

and $\mathbb{P}^2$. Then the unstable locus is the $\mathbb{P}^1$ given via $x_3 = 0$ and (x_1, x_2) is the ideal of the unstable point left out. The unstable $\mathbb{P}^1$ has a point $(1:0:0)$ that moves fastest to the origin, so we may write $\mathbb{P}^1 = pt \coprod \mathbb{A}^1$ where $\mathbb{A}^1$ is somehow over the point $(0:1:0)$ which moves slower toward the origin. The outcome is a nice stratification of the null-cone according to this feature of the points of it.

This introduces Mumford's idea of maximal destabilizing flag which gives a stratification of the unstable set via weight data. This is developed by several people using different approaches

- It is basically combinatorics with polytopes developed by Kemp, Rousseau and Hesselink.
- The Morse theoretic interpretation is developed in Kirwan's thesis under Atiyah, following ideas of Atiyah-Bott.

The stratification data really restricts once again to that of linear actions on $\mathbb{P}^n$.

Remark. Multiplicative group actions have nice behaviors for cohomological purposes. Again consider the above $\mathbb{G}_m$ action on $\mathbb{P}^1$. Our nice stratification $\mathbb{P}^1 = pt \coprod \mathbb{A}^1$. And the same thing with $\mathbb{P}^2 = \mathbb{P}^1 \coprod \mathbb{A}^2 = pt \coprod \mathbb{A}^1 \coprod \mathbb{A}^2$. These $\mathbb{G}_m$ actions on smooth varieties give reasonable decompositions for most cohomology theories, e.g. Bialynicki-Birula decomposition for Chow motive. Equivariant cohomology decomposes even better (Atiyah-Bott ideas, Kirwan's results with have consequences like computation of cohomology of the quotients X^s/G when $X^{ss} = X^s$).

6. Quotient schemes

Mumford was worried about moduli problems so his main motivation of doing GIT was getting quotient schemes. We saw that things work out OK when X is affine or projective and G is reductive. What happens if what we are working with is not affine or projective, for instance maybe we want to keep the data of a line-bundle which is not ample.

The idea is to try to patch affines to get a scheme. So we might have $\{\mathcal{U}_i\}$ covering of X and we construct $\mathcal{U}_i /\!\!/ G = \mathcal{S}pec(k[\mathcal{U}_i]^G)$'s and patch them with universal property of categorical quotient. Then the question is then how in general way would we get these nice $\mathcal{U}_i$'s covering X. And this is somehow answered by defining the problem away! If we have a line bundles $L \to X$ with G-linearization (compatible lift of the G-action from X to L, specified by a character).

Definition 3. When G is reductive and X is a scheme with an invertible sheaf L over it and a linearization of the G action on L, a geometric point $x \in X$

(1) is semistable if there is a section $s \in H^0(X, L^n)$ for some n such that $s(x) \neq 0$, X_s is affine and s is an invariant.

(2) is stable if there is $s \in H^0(X, L^n)$ for some n with $s(x) \neq 0$ affine, and s an invariant and the action of G on X_s is closed (and the isotropy group is zero dimensional).

Note 6.1. L need not be ample, X need not be quasi-projective, but quotient is quasi-projective.

Remark. If G reductive and the action is free in the scheme-theoretic sense on X, then X/G need not be a scheme. But for this to happen you need to make X be a weird funny space, at least X cannot be quasi-projective normal. If G not reductive, it is easy to get scheme-theoretically free actions on X such that X/G is not a scheme (but an algebraic space). (Example: some $\mathbb{G}_a$ action on $\mathbb{A}^n$.)

This linearization technique is an extra flexibility to GIT not exploited until 90's by Thaddeus and Dolgachev-Hu. The latter showed that the are finitely many choices in nice cases for the linearization. This led later on to birational transformations and wall-crossing phenomena. We will not go through them. Here we give two toy models of this extra flexibility.

Example 6.2. $X = \mathbb{A}^n$ and G torus, L a line bundle. X^{uns} given by some coordinate linear subspace encoded in polytopes. $X^{ss}/\!\!/T$ is a toric variety. ⌟

Example 6.3 (Allcock-Carlson-Toledo)**.** $(\mathbb{P}^1)^n$ with the usual action of Sl_2. Then we can pass between chambers associated with different linearizations. We can also study what happens to cohomology (wall-crossing, etc.). This gives new insights to some moduli interpretations. Let's say we want to consider the moduli problem of cubic surfaces. A priori Hodge theory is useless for cubic surfaces. Geometrically you can think of a cubic surface as a $\mathbb{P}^2$ blown-up at 6 points, 5 points corresponding to conic and the 6th point correspond to two tangent lines to conic. So we have 7 points on $\mathbb{P}^1$, 5 of weight 2, 2 of weight 1. So there is a nice linearization we can pick for $(\mathbb{P}^1)^7$ according to these weights. Now there is a nice Hodge theory of local systems due to Deligne-Mostow (the theory of hypergeometric functions) and you can show that often $(\mathbb{P}^1)^n$ have hyperbolic structure (as a quotient of a complex ball with some discrete group) and that the Baily-Borel compactification is the same as the GIT compactification. So we get an induced hyperbolic structure on the moduli of cubic surfaces. ⌟

The are a lot to say about birational transformations and variation of GIT which we will be skipping.

Summary of last sessions: Lecture 4

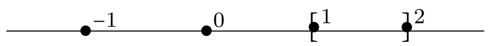
- When G is reductive in characteristic zero. If X is afifine $X \to X/\!\!/G = \mathcal{S}pec(k[X]^G)$ is the categorical quotient and if X is projective there is an open $X^s s \subseteq X$ such that $X \dashrightarrow X^{ss}/\!\!/G$ via $X^{ss} \to X^{ss}/\!\!/G$ is the categorical quotient which is quasi-projective. The basic idea was to embed X, G-equivariently into a projective space with linear representation acting on it at least if X is normal and use functoriality in view of the stratification of $\mathbb{P}^n$ as
$$(\mathbb{P}^n)^s \subseteq (\mathbb{P}^n)^{ss} \subseteq \mathbb{P}^n.$$
Remark (Issue of closed orbits)*.* We should consider closed orbits in the affine cone or in the affine patch not in the projective space itself. For instance in the example of $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, stability is defined by the orbit of x being closed not in $\mathbb{P}^1$ but in the affine cone of $\widehat{x}$. Or in the definition of Mumford, the orbit of x should be closed in $\mathbb{P}^1 \smallsetminus \{xy = 0\}$ (note xy is a degree 2 polynomial and is coming from the fact that x and y are generators of the invariant functions).

- Stability depends on the linearization but is independent of certain choices; e.g. once we embed X via an ample line bundle, embedding it into larger projective spaces $X \hookrightarrow \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ is immaterial since it is the linearization of L versus some tensor power of it $L^{\otimes n}$.

- Hilbert-Mumford numerical criterion: if $\widehat{x}$ is a representative of $x \in X$ in affine cone, then x is unstable iff there is a 1-parameter $\mathbb{G}_n \subset G$ with $0 \in \overline{\mathbb{G}_m.x}$. Another version is

  Theorem 6.1. *If $x \in X$ is given in homogeneous coordinates by $[x_0 : \cdots : x_n] \in \mathbb{P}^n$. x is semistable (resprectively stable) for the action of a maximal torus acting diagonally on $\mathbb{P}^n$ with weights $\alpha_0, \cdots, \alpha_n$ iff the convex hull $Conv\{\alpha_i : x_i \neq 0\}$ contains 0 (respectively contains 0 in its interior).*

- In particular we can identify unstable locus with respect to the maximal torus T and then sweep out by the G-action. This refines the stability stratification by taking the closest point to origin on polytopes that do not contain 0. The basic picture of the example of $\mathrm{diag}(\lambda^2, \lambda, \lambda^{-1})$ is

  for instance in the interval $[1, 2]$ which does not contain 0 we get a $\mathbb{P}^1 = pt \cup \mathbb{A}^1$.

  This has cohomological applications (Atiyah-Bott idea of equivariant perfection) $X = X^{ss} \coprod_i S_i$ then

  $$H^*_G(X) = \oplus_i H^*_G(S_i) \oplus H^*_G(X^{ss}).$$

  In particular in case $X^{ss} = X^s$ then

  $$H^*_G(X^{ss}) = H^*(X^{ss}/G)$$

  (this last space is the moduli space, say vector bundles on a curve, hypersurfaces of degree d in $\mathbb{P}^n$).

  Remark. As far as stability analysis matter, singularities do not complicate GIT but they do affect this sort of cohomological analysis!

- The next thing we said was that Mumford extended the theory by

  (1) patching affine quotients together,

  (2) G-linearization (twists by character of G); classically on the affine patch (semi-invariants).

- We looked at two crucial toy models. (1) Moduli of points on $\mathbb{P}^1$: We vary the ample line bundles. So we consider the G-ample cone. This links a lot of problems in algebraic geometry, e.g. problems about moduli of surfaces, problems on locally symmetric spaces, hyperbolic structures, etc. (2) Toric varieties: when $X = \mathbb{A}^n$ and T is a torus acting on X. Then the unstable set is a union of coordinate linear subspaces and the combinatorics is covered by polytopes. The quotient is a toric variety.

7. Variations and applications

7.1. **Mori dream spaces.** Keel-Hu generalized these examples to Mori dream spaces. In example (2) above we can revert the process and start with a toric variety Y, and look at the toric bundle over it

$$\begin{tikzcd} T \arrow[r] & B \arrow[d] \\ & Y \end{tikzcd}$$

and picking a closure to fill up the unstable points; B is quasi-affine and if its ring of coordinates is finitely generated, then we can take its canonical closure $\mathcal{S}pec\, k[B]$ which is an affine space. The more interesting question is to find T. For that we should look at the finitely-generated Picard group of Y (needed as assumption) and summing over all line bundles there to construct B. Then Y is the GIT quotient and furthermore the Mori chamber decomposition can be identified with the GIT chambers.

Remark. There is a lot of combinatorial control in these examples. This is good to get at *highly rationally connected spaces*. These are spaces with $\pi_i^{\mathbb{A}^1} = 0$ for all i. The philosophy is that free torus actions are $\mathbb{A}^1$-covering spaces in the homotopy category. We may look at the long exact sequence for homotopy fibrations and observe that higher π_i's descent to the quotient so we can talk about higher connectivity properties of these quotient spaces.

7.2. **Applications of linearization.** The extra flexibility in choice of (X, L) is useful. For instance we might have non-separatedness phenomena for X; say we have a not necessarily reductive group $G = R \ltimes U$ (U normal). We might think of taking quotient first with respect to U, $X = Y/U$ may be non-separated as in our example of $Q_2/\mathbb{G}_a$ and now we have a reductive R-action on Y/U. Another case is when X not complete. In this case even if X carries a natural ample line bundle $L \to X$ ample, it's completion, $\overline{X}$, may not. Suppose H acts on X. Then we have to pass to a larger group G, an look at $G *_H X$. This will now have also an ample line bundle. In many examples semi-ample line bundles are useful (c.f. Ressayre's Inventiones paper of 2010 and Knutson-Tao results on Horn conjecture).

7.3. **Theory of birational cobodisms.** What if do not have a G-action? Think of X as a quotient and use higher dimensional space with G-action. From birational geometry we have the motivating example called the Atiyah flip, namely $\mathbb{A}^{\ell+m}$ with $(1, \cdots, 1, -1, \cdots, -1)$-weight action of $\mathbb{G}_m$. Then $\mathbb{A}^{\ell+m} /\!\!/ \mathbb{G}_m$ is the affine cone over the Segre embedding of projective spaces $\mathbb{P}^{\ell-1} \times \mathbb{P}^{m-1}$. We also can take two geometric quotients for the positive weight and negative weight pieces and a diagram

$$\begin{tikzcd} & \mathbb{A}^{\ell+m} \arrow[dl] \arrow[d, two heads] \arrow[dr] & \\ B_-/\mathbb{G}_m \arrow[rr, dashed, bend right] & \mathbb{A}^{\ell+m} /\!\!/ \mathbb{G}_m & B_+/\mathbb{G}_m \end{tikzcd}$$

where the flip replaces $\mathbb{P}^{\ell-1}$ with $\mathbb{P}^{m-1}$ assuming $\ell, m \geq 2$. We can zig-zag like this and achieve Wlodarczyk's theory of birational cobordism which is crucial in proof of weak factorization. The weak factorization briefly says that if working in characteristic zero and smooth proper schemes, then a birational $X_1 \dashrightarrow X_n$ factors through finitely many blow-ups and blow-downs with smooth centers

$$\begin{tikzcd}
& Y_1 \arrow[dl] \arrow[dr] & & Y_2 \arrow[dl] \arrow[dr] & & \cdots & Y_n \arrow[dr] & \\
X_1 \arrow[rr, dashed] & & X_2 \arrow[rr, dashed] & & X_2 & \cdots & & X_n
\end{tikzcd}$$

The strong factorization replaces the zig-zag with a sequence of all blow-ups and then all blow-downs. (smooth centers is crucial; with GIT we get lots of toric singularities). The $\mathbb{G}_m$ actions are used to shift within a birtaional class.

7.4. **Analogies with algebraic topology.** All of GIT and applications boil down to linearization of the $\mathbb{G}_m$ actions and this is basically the theory of $\mathbb{A}^1$-covering spaces, plus combinatorics! So although GIT has a few additional complications, it is now in a similarly complete phase as the theory of covering spaces in algebraic topology. I we pursue this analogy with topology further, we may ask what happens for homotopy. Clearly we want $X \times \mathbb{A}^1 \sim X$.

Recall the question of $X \times \mathbb{A}^1 = Y \times \mathbb{A}^1$ then $X \cong Y$? for $\log Kod \geq 0$ this works OK, and for other nearly rational spaces it does not and we had an example of this. But what if we ask the same question but merely for birational equivalence.

Theorem 7.1 (B, C-T, S, S-D). *Birational version of this cancellation is false: there exists a smooth complex conic bundle over $\mathbb{P}^2$ such that X is not rational but stably rational. $X \times \mathbb{P}^2$ is birationally equivalent to $\mathbb{P}^5$ but X is not birationally equaivalent to $\mathbb{P}^3$.*

So homotopy moves between nearly birational classes.

Recall the group of Euclidean motions $\mathcal{O}(2) \ltimes \mathbb{R}^2$. In projective spaces we have the group of automorphisms $\mathbb{P}\mathrm{GL}_n$ acting on. What if we consider the weighted projective spaces (or toric varieties, or sphericals, etc.). The group of automorphisms of $\mathbb{P}(1,1,2)$ for instance is $\mathrm{GL}_2\, n \times \mathrm{GL}_1 \ltimes (\mathbb{C}^*)^2$. When looking at configurations of hypersurfaces in projective spaces then we will have to worry about these actions. (For example look at the degree four curves, CY $\Rightarrow$elliptic curves.) Studying families of hypersurfaces in toric varieties is a general tool for construction of Calabi-Yau moduli spaces.

Let X be normal affine and G unipotent acting freely. The first thing we know is that X/G is an algebraic space in sense of Artin. We have to ask ourselves is this a scheme? This is sort of a stability test in sense of Hilbert-Mumford.

- If yes. Then it is quasi-affine. Is it affine? This is a test of vanishing of Lie-algebra cohomology; $H^*(\mathfrak{g}, k[x]) = 0$ iff X/G is affine.

  - If yes. Then $X \to X/G$ admits a section. The G-action is then conjugate to the trivial action. And since unipotent groups are isomorphic to affine spaces, we get an isomorphism $X \cong X/G \times \mathbb{A}^k$.
  - If no, does there exists a canonical embedding $\varphi : X/G \hookrightarrow \overline{X/G}$ into an affine closure with complement codimension greater than 1. This is Hilbert's fourteenth problem is disguise.
- If no. This can happen: take $\mathbb{G}_a$-representation, look at a sufficiently generic closed subvariety X of intermediate codimension.

7.5. **Case of non-finitely generated ring of invariants.** As an example, take an elliptic curve Y. Look at universal abelian extensions: $\mathbb{G}_a$-torsors on Y. Y is an algebraic group hence quasi-projective and in particular has an ample line bundle. Look at the $\mathbb{G}_m$-torsor, X inside that ample line bundle. This latter is a 3-dimensional quasi-affine variety. It does not admit a codimension two complement. The point is that $k[X]$ is not finitely-generated. To make a $\mathbb{G}_a$ action one thing we can do is to consider the embedding $X \hookrightarrow \overline{X}$ such that $\overline{X}$ is normal. Then $\overline{X}$ is an S_2-variety in the sense of Serre. So we can cut out the complement by a pair of functions, $\overline{X} \smallsetminus S = \{f = 0, g = 0\}$. Then we can take the fiber product

$$\begin{tikzcd}
\overline{X}_{\mathbb{A}^2} \operatorname{Sl}_2 \arrow[r] \arrow[d] & \mathrm{SL}_2 \arrow[d, "\text{invariant functions}"] \\
\overline{X} \arrow[r, "(f,g)"'] & \mathbb{A}^2
\end{tikzcd}$$

which has a $\mathbb{G}_a$ action on it and everything is happy! Remember that $\operatorname{Sl}_2 \twoheadrightarrow \mathbb{A}^2 \smallsetminus \{0\} = \operatorname{Sl}_2 / \mathbb{G}_a$. We can now recover X as a quotient $\overline{X}_{\mathbb{A}^2} \operatorname{Sl}_2 / \mathbb{G}_a$.

7.6. **Applications to moduli of bundles.** In topology, there is a prefect theory of moduli of bundles. Basically if X is a manifold, and $V_k(X)$ is the set of isomorphism classes of rank k topological vector bundles. We have a bijection $V_k(X) \leftrightarrow [X, B\operatorname{Gl}_k]$. We can also work with principal G-bundles and get a correspondence with $[X, BG]$.

In algebraic geometry even in the affine case, it is very hard to find $V_k(\mathbb{A}^n)$. It was solved by Serre and Quillen-Suslin. If X is smooth and affine Lindel pushed the techniques further to show there is a canonical bijection

$$V_k(\mathbb{A}^1 \times X) \leftrightarrow V_k(X).$$

So we might hope that this homotopy invariance extends further and indeed if X is a smooth affine variety we have

$$V_k(X) \leftrightarrow [X, BGL_k]_{\mathbb{A}^1}.$$

To get a theory of moduli of vector bundles for projective varieties we may pass to the quasi-affine cover in the cone and then to something affine.

$$\begin{tikzcd} A \arrow[d, "\mathbb{A}^1\text{-cover}"'] & \text{affine} \arrow[l, hook', "\text{homotopy equiv.}"] \\ P & \end{tikzcd}$$

The only hard part is the step of getting the affine space. For this we need to understand how the twists and pullbacks change the (moduli of) vector bundles.

Theorem 7.2 (Asok-Doran)**.** *Given a triple of integers* $(n \geq 6, \ell, m)$ *then there exists a scheme-theoretically free* $\mathbb{G}_a$*-action on* $\mathbb{A}^n$ *such that* $\mathbb{A}^n/\mathbb{G}_a$ *is an affine variety and* $V_k(\mathbb{A}^n/\mathbb{G}_a)$ *has at least* ℓ*-dimensional moduli for every sufficiently high* k*. We can do the same thing for families of* m *pairwise non-isomorphic such objects.*

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