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RINGEL-HALL ALGEBRAS AND APPLICATIONS TO MODULI

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1. Motivations

Lecture 1

The general setup is that we have an abelian category, $\mathcal{A}$, and we want to study moduli space of isomorphism classes of objects in this! We fix some *degree* d and have the notion of *stability* and then we have, $\mathcal{M}_d^{st}(\mathcal{A})$, the moduli of stable objects or variants thereof. A typical variant is the moduli of poly-stable objects (e.g. poly-stable representations) or objects together with additional structure (e.g. objects that can be written as quotients).The point is that there exists a noncommutative algebra $\mathcal{H}_{\mathcal{A}}$ encoding quantitative information about $\mathcal{M}_d^{st}(\mathcal{A})$ (something like Betti numbers, Euler characteristics, etc.). The important task is to ask the right question for this algebra to answer!

There are two main situations to which this strategy has been applied to:

(1) $\mathcal{A}$ of cohomological dimension ≤ 1 (i.e. all higher Ext groups vanish). Main examples:

  (a) $Coh(C)$, coherent sheaves on a smooth projective curve;

  (b) $\operatorname{Rep} Q$, representations of a quiver (the case we will consider).[1]

(2) $\mathcal{A}$ is an abelian (Joyce-Song), triangulated, A_∞ or whatever (Kontsevich-Soibelman) CY_3 category.

In the $\operatorname{Rep} Q$ case we in essence only do linear algebra and as a result all the basic qualifying information (connectedness, irreducibility, dimension, etc.) are absolutely clear there. Then we can go on to derive quantitative information, i.e. invariants of the moduli spaces.

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[1]If this is not already interesting to you then whenever we have representation of a quiver V just replace it by $\mathcal{E}$, a smooth projective surface and all the basic properties will go through because of cohomological dimension ≤ 1.

2. Quivers and their representations

2.1. **Motivation.** [GIT quotients] One classical action of a reductive group on some affine space is the action of $\mathrm{Gl}_d(\mathbb{C})$ on $M_{d\times d}(\mathbb{C})$ via conjugation. We can make things more complicated by considering

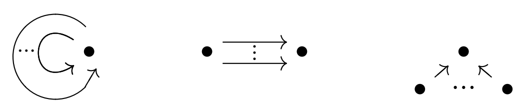
(1) simultaneous action of this group on $M_{d\times d}(\mathbb{C})^m$ (pops up for instance in moduli problem of vector bundles on curves).
(2) action of $\mathrm{Gl}_d \times \mathrm{Gl}_e$ on $M_{d\times e}(\mathbb{C})^m$ via base change (pops up when considering vector bundles on $\mathbb{P}^2$)
(3) action of PGL_d on linear subspaces of projective space $(\mathbb{P}^{d-1})^m$.

These are the situations that interesting moduli problems get reduced to, and beside that they are already interesting problems of linear algebra (e.g. understanding normal forms of simultaneous actions of general linear groups is an unsolved linear algebra problem). It is interesting to work out the quotients and their cohomologies in the above case. Now the quiver representations serve as a unified language for such problems.

3. Basics of quivers

Definition 1. A quiver Q is a (finite) oriented graph; Let I be the set of vertices and $\{\alpha : i \to j\}$ the set of arrows.

Example 3.1.



$\lrcorner$

Definition 2. Let K be any field. A K-representation V of Q is

$$V = ((V_i)_{i\in I}, (V_\alpha : V_i \to V_j)_{\alpha:i\to j})$$

where V_i's are finite dimensional K-vector spaces and the maps are K-linear maps. A morphism of representations V and W, $f : V \to W$ is a tuple $(f_i : V_i \to W_i)_i$ for each vertex $i \in I$ such that all diagrams

$$\begin{tikzcd} V_i \arrow[r, "V_\alpha"] \arrow[d, "f_i"] & V_j \arrow[d, "f_j"] \\ W_i \arrow[r, "W_\alpha"] & W_j \end{tikzcd}$$

commute. The composition is defined in the obvious way.

Fact: This defines an abelian category $\mathcal{Rep}_K Q$. It is a finite length category (i.e. the objects don't produce infinite filtrations).

Example 3.2. A subrepresentation is a tuple $(U_i \subset V_i)_{i\in I}$ such that $V_\alpha(U_i) \subset U_j$ for all α (compatible with everything). Up to isomorphism any quiver representation is just $V = ((K^{d_i}), A_\alpha \in M_{d_j \times d_i}(K))$ and this is isomorphic to $W = ((K^{d_i}), B_\alpha \in M_{d_j\times d_i}(K))$ if and only if $(B_\alpha)_\alpha$ arises from $(A_\alpha)_\alpha$ by simultaneous base change in all K^{d_i}; i.e. automorphisms $(g_i)_i$ exist such that for all $\alpha : i \to j$ we have $B_\alpha = g_j A_\alpha g_i^{-1}$. $\lrcorner$

Example 3.3. For the quiver

$$\begin{tikzcd} \bullet \arrow[loop left] \end{tikzcd}$$

a representation is the just a vector space and an endomorphism: (V, φ). A subobject is an invariant subspace. Up to isomorphism replace (V,φ) by K^d and a matrix A, and $A \sim B$ if and only if $B = gAg^{-1}$ for some automorphism g. In particular, over $\mathbb{C}$ the Jordan canoncial forms classify our obejcts!

Continuous invariants: eigenvalues; Discrete invariants: sizes of Jordan blocks. $\lrcorner$

Example 3.4. For the quiver

$$\bullet \longrightarrow \bullet$$

a representation is $f : V \to W$. Up to isomorphism $A, B : K^d \to K^e$ are isomorphic $A \sim B$ if and only if $\exists g, h$ such that $B = hAg^{-1}$. So get classification by elementary row and column operations $\begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}$.

Only 3 discrete invariants: $\dim V$, $\dim W$ and $\operatorname{rk} f$. $\lrcorner$

Example 3.5. Let's consider the quiver

$$\begin{tikzcd}
& & \bullet & & \\
\bullet \arrow[urr] & & & & \bullet \arrow[ull] \\
& \bullet \arrow[uur] & & \bullet \arrow[uul] &
\end{tikzcd}$$

And we make things very special by considering the following configuration of vector spaces

$$\begin{tikzcd}
& & K^2 & & \\
K \arrow[urr] & & & & K \arrow[ull] \\
& K \arrow[uur] & & K \arrow[uul] &
\end{tikzcd}$$

Generically such a representation is isomorphic to

$$\begin{tikzcd}
& & K^2 & & \\
K \arrow[urr, "\binom{1}{0}"] & & & & K \arrow[ull, "\binom{\lambda}{\mu}"'] \\
& K \arrow[uur, "\binom{0}{1}"] & & K \arrow[uul, "\binom{1}{1}"'] &
\end{tikzcd}$$

where $\binom{\lambda}{\mu}$ is defined up to scalar multiplication. So generically have a continuous invariant, namely the cross-ratio:

$$\mathbb{P}^1 \cong (\mathbb{P}^1)^4_{sst}/\,\mathrm{PGL}_2\,.$$

⌟

Theorem 3.1 (Gabriel). *$\mathcal{Rep}_K Q$ has only finitely many indecomposable representations / $\cong$ if and only if $|Q|$ is a disjoint nion of A_n, D_n, E_6, E_7, E_8. In fact these are in one to one correspondence with Φ^+ the positive root system, via $V \mapsto \underline{\dim} V \in \Phi^+ \subset \mathbb{N}I$.*

Example 3.6. D_4 has α_i's , $\alpha_1 + \alpha_i$'s, $\alpha_1 + \alpha_i + \alpha_j$'s $i \neq j$, $\alpha_1 + \cdots + \alpha_4$ and $2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$. Check there are 12 indecomposables. ⌟

For further theory of representations of quivers refer to homepage of Crawyley-Boevey (Leeds): Lectures on representations of quivers.

Note 3.7. In some cases we do not have a classification of isomorphism classes of objects in the category, namely if we have lots of loops or lots of arrows. There is a whole theory related to distinguishing such cases. For instance you might ask which quivers only have discrete invariants. The necessary and sufficient condition is that $|Q|$ is a disjoint union of the Dynkin diagrams A, D and E where we get a moduli space consisting only of points! You have *harmless* continuous invariants if and only if the quiver is a disjoint union of extended Dynkin diagrams. We will not be going through this story since our aim is to get interesting moduli spaces to the contrary! ⌟

4. Facts on $\mathcal{Rep}_K Q$

4.1. There is an equivalence of categories $\mathcal{Rep}_K Q$ and $\mathcal{Mod}\, KQ$ modules over the path algebra KQ.

Remark. $D^b(\mathbb{P}^n) \cong D^b(\bigwedge^\bullet \mathbb{C}^n)$ is an instance of a higher dimension analogous equivalence of categories.

4.2. $K_0(\mathcal{Rep}_K(Q)) \to \mathbb{Z}I = \Gamma$ via $[V] \mapsto \underline{\dim} V = (\dim_K V_i)_{i \in I}$ is a well-defined map, because dimensions are additive along short exact sequences. $\mathbb{Z}I$ is the numerical version of the Grothendieck group!

▸ **Exercise 1.** The above mapping is an isomorphism if Q has no oriented cycles.

Solution. In this case the only irreducible representations are the ones with only a one-dimensional vector space assigned to one vertex and all other vertices and maps are trivial. ■

4.3. $\mathcal{Rep}_K(Q)$ is a category of cohomological dimension 1, i.e. $\mathrm{Ext}^{\geq 2} \cong 0$

4.4. $\dim \mathrm{Hom}(V,W) - \dim \mathrm{Ext}^1(V,W) = \langle \underline{\dim} V, \underline{\dim} W\rangle$ is the Euler form computing Euler characteristic, and is concretely given by

$$\langle d, e\rangle := \sum_{i\in I} d_i e_j - \sum_{\alpha: i\to j} d_i e_j.$$

To derive the above formula we introduce certain projective resolutions of the representations:

Remark. If A is finite dimensional algebra over $\mathbb{C}$ then A considered as a left module on itself is projective. So we have enough projectives always over $\mathcal{M}od\,\mathbb{C}Q$; the first step of projective resolution is then going to be $A \otimes_{\mathbb{C}} M \to_A M \to 0$.

Theorem 4.1 (Standard resolution)**.** *For every $i \in I$, define $(P_i)_j = \langle$ paths ω from i to j $\rangle$ and for any $\alpha : j \to k$ get a morphism $(P_i)_j \overset{(P_i)_\alpha}{\to} (P_i)_k$ via $w \mapsto w\alpha$. Then $\oplus_{i\in I} P_i$ is isomorphic to the path algebra $\mathbb{C}Q$ viewed as a left module over itself. So we always have a resolution as follows:*

$$0 \to \oplus_{\alpha: i\to j} P_j \otimes V_i \to \oplus_{i\in I} P_i \otimes_{\mathbb{C}} V_i \to V \to 0$$

the last map is $w \otimes v \mapsto V_w(v)$ and the former map is $w \otimes v \mapsto w \otimes V_\alpha(v)$.

Note 4.1. The structure of these projective modules: any such P is a direct summand of $(\mathbb{C}Q)^m$ and we know that $\oplus_{i\in I} P_i \cong \mathbb{C}Q$ hence P is a direct summand of $\oplus_{i\in I} P_i^n$ and these are hence the simple algebras! ⌟

4.5. There is something like Serre duality!

$$\mathrm{Ext}^1(V,W) \cong \mathrm{Hom}(W, \tau V)^\vee$$

except when V, W is injective or projective. τ is defined via

$$\tau(X) := \mathrm{Ext}^1_{\mathbb{C}Q}({}_{\mathbb{C}Q}X, {}_{\mathbb{C}Q}\mathbb{C}Q)^\vee,$$

but it is not clear why this is itself a representation. It is neither easy to compute the τ's but here is an

Example 4.2. Let Q be $1 \to 2 \to \cdots \to n$. The root system is

$$\Phi^+ = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j : 1 \le i \le j \le n\}.$$

Let $V_{[i,j]}$ be the (indecomposable) representation corresponding to

$$0 \to \cdots \to 0 \to \mathbb{C} \overset{\mathrm{id}}{\to} \cdots \overset{\mathrm{id}}{\to} \mathbb{C} \to 0 \to \cdots \to 0.$$

Then $\tau V_{[i,j]}$ will be $V_{[i+1,j+1]}$ when $i, j < n$ and 0 if $i, j = n$. ⌟

▸ **Exercise 2.** Work out all the dimensions of Homs and Exts for the quiver $\bullet \to \bullet$.

5. Moduli of representations

We have an abelian category that is very much like the category of coherent sheaves on a smooth projective curve and we imitate all the necessary notions to construct moduli spaces. In contrast to the vector bundles on curves (rank and degree) there is absolutely no canonical choice of stability. Rank is analogous to the sum of dimensions of the vector spaces appearing in the representation. Our replacement for degree will be a choice of an additive function $\Theta : \mathbb{Z}I \to \mathbb{Z}$ (so $\Theta(d) = \sum_{i\in I} \Theta_i d_i$). Now we can define slope:

$$\begin{aligned} \mu : \mathbb{N}I - \{0\} &\to \mathbb{Q} \\ d &\mapsto \Theta(d)/\dim d \end{aligned}$$

where $\dim d := \sum_{i\in I} d_i$ is the replacement for the rank. And define for any $0 \neq V \in \mathcal{Rep}_K Q$ its slope as the slope of its dimension vector:

$$\mu(V) := \mu(\underline{\dim} V).$$

Definition 3. V is (semi-)stable if and only if for any $0 \neq U \subsetneq V$ have $\mu(U) \underset{(\leq)}{<} \mu(V)$ and is polystable whenever V is isomorphic to direct sum of stables of same slope.

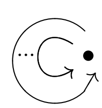
All usual properties of (semi)-stables hold. For instance the set of all objects

$$\{V : \text{semistable}, \mu(V) = \mu\}$$

define an abelian subcategory, have HN filtrations, etc. The only question is whether we are getting anything interesting since everything is depending on our choice of Θ. In general this choice of stability condition is a subtle one, but

▸ **Exercise 3** (Important)**.**

(1) For the quiver



all Θ are equivalent to 0 (in the sense that they yield the same (semi-)stable objects).

(2) For the quiver

$$\bullet \;\substack{\longrightarrow \\ \vdots \\ \longrightarrow}\; \bullet$$

all Θ are equivalent to one of $(0,0)$, $(0,1)$ and $(1,0)$ where only the last kind is interesting to us.

6. Construction of the moduli

We want to apply GIT to create $M_d^{\Theta-poly}(Q)$. One can work out that the notion of semi-stability here is the same as the one used in GIT. Therefore,

$$M_d^{\Theta-poly}(Q) = \left(\oplus_{\alpha:i\to j} M_{d_j\times d_i}(\mathbb{C})\right)^{sst} /\!\!/ \prod_{i\in I} \mathrm{Gl}(\mathbb{C}^{d_i})$$
$$= \mathcal{P}roj(\mathbb{C}[\oplus_\alpha M_{d_j\times d_i}]_{\chi_\Theta}^{\prod_i \mathrm{Gl}(\mathbb{C}^{d_i})})$$

(the group action obviously is reductive) and

$$\chi_\Theta : \prod_i \mathrm{GL}(\mathbb{C}^{d_i}) \to \mathbb{C}^*$$
$$(g_i)_i \mapsto \prod_{i\in I} \det(g_i)^{\Theta(d)-\dim d.\Theta_i}$$

For convenience we fix a new notation and that is $R_d(Q)$ which is the parameter space of all representations of all types: Lecture 2

$$R_d(Q) = \oplus_{\alpha:i\to j} \mathrm{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$$

and has the action of $G_d = \prod_{i\in I} \mathrm{GL}(\mathbb{C}^{d_i})$ on it by $(g_i)_i.(V_\alpha)_\alpha = (g_j V_\alpha g_i^{-1})_{\alpha:i\to j}$. This vector space appeared above: $R_d(Q)^{sst}/\!\!/G_d := \mathcal{P}roj(\oplus_{n\geq 0}\mathbb{C}[R_d(Q)]^{G_d,\chi_\Theta^n})$. There is a projective morphism to $\mathcal{M}_d^{0-pst} = \mathcal{S}pec(\mathbb{C}[R_d(Q)]^{G_d})$ and this latter is a point if Q has no oriented cycles by an exercise above.

We shall record the following facts:

6.1. $\mathcal{M}_d^{\Theta-pst}(Q)$ is projective over the affine scheme $\mathcal{M}_d^{0-pst}$.

6.2. There is an open locus $\mathcal{M}_d^{\Theta-st}(Q)$ which is smooth and irreducible since we are taking geometric quotient of a smooth irreducible affine universal bundle.

6.3. We can use the Euler form to comment on the dimension of it also:

$$\dim \mathcal{M}_d^{\Theta-st}(Q) = 1 - \langle d, d\rangle$$

if nonempty.

Remark. This Euler form was of homological meaning but now we see that it also appears in the moduli, so the moduli has some homological information in it as well.

For later applications (in wall-crossing) we also need a variant of these moduli spaces: Instead of merely looking at vector bundles, $\mathcal{E}$, we can choose projections $\mathcal{F} \twoheadrightarrow \mathcal{E}$ as if in the quot scheme construction. So we shall choose another dimension vector $n \in \mathbb{N}I$ and consider pairs (V, f) such that V is Θ-semistable and $f = (f_i : \mathbb{C}^{n_i} \to V_i)_{i\in I}$. We require f to be avoiding bad subrepresentations; i.e. if $U \subseteq V$ and $\mathrm{im}\, f \subseteq U \subseteq V$ then $\mu(U) < \mu(V)$. And we are looking at them up to the action of G_d. Now we take the geometric quotient which we call $Hilb^{\Theta}_{d,n}(Q)$, There is a projective morphism $Hilb^{\Theta}_{d,n}(Q) \xrightarrow{\pi} \mathcal{M}_d^{\Theta-pst}(Q)$ which can be considered as is a *non-commutative Hilbert-Chow morphism*. Over the stable locus $\mathcal{M}_d^{\Theta-st}(Q)$, π is just a projective space fibration.

6.4. If d is coprime then for generic choice of Θ, semi-stability and stability coincide so in this case we know that $\mathcal{M}_d^{pst} = \mathcal{M}_d^{st}$ and this is a smooth projective scheme (over our obvious affine space). Then $Hilb^{\Theta}_{d,n}(Q) = \mathbb{P}(\oplus V_i^{n_i})$ is the projectivization of a direct sum of tautological bundles.

7. Hall algebras

In this section the base field is a finite field $k = \mathbb{F}_q$ and we are looking at $\mathcal{R}ep_k(Q)$. We present a completed version of the Hall algebras. As a set such an algebra is

$$\mathcal{H}_k(Q) := \prod_{d \in \mathbb{N}I} \mathbb{Q}^{G_d(k)}(R_d(Q)(k)),$$

the set of arbitrary $G_d(k)$-invariant functions $f : R_d(Q)(k) \to \mathbb{Q}$. Note that the domain is just a finite set! For $f : R_d(Q)(k) \to \mathbb{Q}$ and $g : R_e(Q)(k) \to \mathbb{Q}$ the product is defined via

$$(f * g)(V) = \sum_{U \subseteq V} f(U)g(V/U)$$

where V ranges over elements of $R_{d+e}(Q)(k)$. V/U makes sense since f and g are invariant functions, so if we make a choice of basis for V/U we get an element of $R_e(Q)(k)$ and the above sum is well-defined.

Fact: $\mathcal{H}_k(Q)$ is an associative $\mathbb{N}I$-graded $\mathbb{Q}$-algebra with unit

$$\begin{aligned} 1 : R_0(Q)(k) &\to \mathbb{Q} \\ pt &\mapsto 1 \end{aligned}$$

$\mathcal{H}_k(Q)$ has a (topological) basis consisting of the characteristic functions 1_V of orbits (i.e. isomorphism classes in $\mathcal{R}ep_k(Q)$).

$$1_V * 1_W = \sum_X \#\{U \subset X : U \cong V, X/U \cong W\}.1_X$$

▸ **Exercise 4** (Important)**.** For the quiver $\bullet \to \bullet$ show that $\mathcal{H}_k(Q)$ has as topological basis $X^k Y^\ell Z^m$ for $k, \ell, m \geq 0$ and has defining relations

$$Y = ZX - XZ, \quad YX = q.XY \quad \text{ and } \quad ZY = qYZ$$

where $X = 1_{k \xrightarrow{0} 0}$, $Y = 1_{k \xrightarrow{1} k}$ and $Z = 1_{0 \xrightarrow{0} k}$.

Definition 4. Some functions of interest to us are

$$\mathbf{1} : V \mapsto 1,$$

and for any $\mu \in \mathbb{Q}$,

$$\mathbf{1}_\mu(V) = \begin{cases} 1 & V \text{ is semistable of slope } \mu \\ 0 & \text{otherwise or if } V \text{ is the zero representation} \end{cases}.$$

Lemma 1 (Harder-Narasimhan recursion)**.** *In $\mathcal{H}_k(Q)$ we have*

$$\mathbf{1} = \overleftarrow{\prod_{\mu \in \mathbb{Q}}} \mathbf{1}_\mu.$$

Proof. We compute the right hand side on V. This is

$$\begin{aligned}
\overleftarrow{\prod_{\mu \in \mathbb{Q}}} \mathbf{1}_\mu(V) &= \sum_{\mu_1 > \cdots > \mu_s} (\mathbf{1}_{\mu_1} * \cdots * \mathbf{1}_{\mu_s})(V) \\
&= \sum_{\mu_1 > \cdots > \mu_s} \sum_{0 = V_0 \subset \cdots \subset V_s = V} (\mathbf{1}_{\mu_1}(V_1/V_0) * \cdots * \mathbf{1}_{\mu_s}(V_s/V_{s-1})) \\
&= \#\{0 = V_0 \subset \cdots \subset V_s = V : \\
&\qquad \text{all factors semistable and } \mu\text{'s are strictly decreasing}\} \\
&= \# \text{ of Harder-Narasimhan filtrations} = 1
\end{aligned}$$

□

Lemma 2. $\mathbf{1}_\mu^{-1} = \sum_V \gamma_V . 1_V$ *where* $\gamma_V = 0$ *unless* V *is polystable of slope* μ *and in this case* $V = \oplus_{U:stable} U^{m_u}$ *and* γ_V *only depends on* $\underline{\dim}\ U$, m_U *and* $|End(U)|$ *the cardinality of the endomorphism ring of the stables (this is a certain finite extension of* $\mathbb{F}_q$ *in this context).*

Remark. From these lemmas it seems that the Hall algebra *knows something* about the stables!

8. Integration on Hall algebras

Lecture 3

Definition 5. $\mathbb{Q}_q[\![I]\!] = \mathbb{Q}[\![t_i : i \in I]\!]$ where q is number of elements in our underlying field. It looks like a power series ring but is slightly noncommutative: let us use the notation $t^d = \prod t_i^{d_i}$ then,

$$t^d . t^e = q^{-\langle e,d \rangle} t^{d+e}$$

and define

$$\begin{aligned}
\int : \mathcal{H}_k(Q) &\to \mathbb{Q}_q[\![I]\!] \\
f &\mapsto \sum_{[V]} \frac{f(V)}{|\operatorname{Aut}(V)|} t^{\underline{\dim} V}
\end{aligned}$$

here $[V]$ ranges over all isomorphism classes.

Example 8.1.

$$\begin{aligned}\int \mathbf{1} &= \sum_{V/isom} \frac{1}{|\operatorname{Aut} V|} t^{\underline{\dim} V} \\ &= \sum_{d\in\mathbb{N}I} \frac{|R_d(\mathbb{Q})(k)|}{|G_d(k)|}.t^d \\ &= \sum_{d\in\mathbb{N}I} \frac{q^{-\langle d,d\rangle}}{\prod_{i\in I}\prod_{k=0}^{d_i-1}(1-q^{-k})} t^d.\end{aligned}$$

The final result is explicit and is independent of the quivers and representations! ⌟

However for

$$\int \mathbf{1}_\mu = 1 + \sum_{d:\mu(d)=\mu} \frac{|R_d^{sst}(Q)(k)|}{|G_d(k)|} t^d.$$

we cannot go further to an explicit formulation. But after all have $\int \mathbf{1} = \int \prod_\mu \mathbf{1}_\mu$. But to use this need we to be able to commute the integration and product. In other words we want to show that,

Proposition 1. $\int : \mathcal{H}_k(Q) \to \mathbb{Q}_q[\![I]\!]$ *is an algebra homomorphism.*

Proof. We have to prove $\int(\mathbf{1}_V * \mathbf{1}_W) = \int \mathbf{1}_V * \int \mathbf{1}_W$. This is equivalent to

$$\sum_X \#\{U \subseteq X : U \cong V, X/U \cong W\} \frac{1}{|\operatorname{Aut}(X)|} t^{\underline{\dim} X} = \frac{q^{-\langle \dim W, \dim V\rangle}}{|\operatorname{Aut} V||\operatorname{Aut} W|} t^{\dim V + \dim W}.$$

Consider the set of all short exact sequences $V \to X \to W$ and call it $\mathcal{S}$. Two short exact sequences are equivalent if there is an isomorphism $X \cong X'$ such that

$$\begin{array}{ccccc} V & \longrightarrow & X & \longrightarrow & W \\ \| & & \| & & \| \\ V & \longrightarrow & Y & \longrightarrow & W \end{array}$$

commutes. Then as a set $\operatorname{Ext}^1(W,V)_X$ is the set of extensions $(W \to X \to V)$ where the middle term is isomorphic to X, up to the above equivalence of extensions.

On the set $\mathcal{S}$ the groups $\operatorname{Aut} V$, $\operatorname{Aut} W$ and $\operatorname{Aut} X$ act. It is clear that $\operatorname{Aut} V \times \operatorname{Aut} W$ acts freely but simple diagram chasing shows that $\operatorname{Aut} X$ acts with stabilizer $1 + \alpha \operatorname{Hom}(W,V)\beta$. If we mod out by $\operatorname{Aut} X$ then we get $\operatorname{Ext}^1(W,V)_X$ and if we mod out by $\operatorname{Aut} V \times \operatorname{Aut} W$ we get our set of interest

$$\{U \subseteq X : U \cong V, X/U \cong W\}.$$

So

$$\begin{aligned}\mathrm{LHS} &= \sum_X \frac{\#\operatorname{Ext}^1(w,V)_X |\operatorname{Aut} X|}{|\operatorname{Aut} V||\operatorname{Aut} W||\operatorname{Hom}(W,V)|}\frac{1}{|\operatorname{Aut} X|} t^{\underline{\dim} V + \underline{\dim} W} \\ &= \frac{t^{\underline{\dim} V + \underline{\dim} W}}{|\operatorname{Aut} V||\operatorname{Aut} X||\operatorname{Hom}(W,V)|}\sum_X |\operatorname{Ext}^1(W,V)_X| \\ &= \frac{t^{\underline{\dim} V + \underline{\dim} W}}{|\operatorname{Aut} V||\operatorname{Aut} X|}\frac{|\operatorname{Ext}^1(W,V)|}{|\operatorname{Hom}(W,V)|} \\ &= \frac{t^{\underline{\dim} V + \underline{\dim} W}}{|\operatorname{Aut} V||\operatorname{Aut} X|} q^{\dim \operatorname{Ext}^1(W,V) - \dim \operatorname{Hom}(W,V)} = \mathrm{RHS}\end{aligned}$$

□

Note 8.2. The previous calculation works in any abelian category, however the crucial point is that the previous proposition used the assumption of cohomological dimension 1 when we wanted to show that $\dim \operatorname{Ext}^1(W,V) - \dim \operatorname{Hom}(W,V)$ is the Euler character and therefore factors through the Grothendieck group. Luckily, you might expect that something works in the CY_3 setting because there we have a duality between Ext^i and Ext^{3-i}.

9. An application: Betti numbers and number of rational points of $\mathcal{M}_d^{\Theta-poly}(Q)$

Start with the identity

$$\mathbf{1} = \overleftarrow{\prod_{\mu\in\mathbb{Q}}} \mathbf{1}_\mu$$

in the Hall algebra $\mathcal{H}_k(Q)$. Integrate both sides to get an identity in $\mathbb{Q}_q [\![I]\!]$.

$$\sum_{d\in\mathbb{N}I} \frac{q^{-\langle d,d\rangle}}{\prod_{i\in I}\prod_{k=0}^{d_i-1}(1-q^{-k})} t^d = \overleftarrow{\prod_{\mu\in\mathbb{Q}}}\Big(1 + \sum_{d:\mu(d)=\mu} \frac{|R_d^{sst}(Q)(k)|}{|G_d(k)|} t^d\Big)$$

Comparing coefficients gives a recursive expression for $\frac{|R_d^{sst}(Q)(k)|}{|G_d(k)|}$ in terms of coefficients on the left hand side.

Back to geometry of moduli spaces: We want to know Betti numbers in singular cohomology of our moduli spaces and what we already have found is a counting of the number of rational points over finite fields. The relation of the two is given by Deligne's theorem/Weil's conjecture for any smooth projective variety.

Assume that d is coprime and Θ is generic. Then $\mathcal{M}_d^{\Theta-poly}(Q) = \mathcal{M}_d^{\Theta-st}(Q)$ and $R_d(Q)^{sst} = R_d(Q)^{st}$ so we have a PG_d principal bundle $R_d(Q)^{st} \to \mathcal{M}_d^{\Theta-st}(Q)$.

Another fact we will use is that there exists a scheme $X/\mathcal{S}pec\,\mathbb{Z}$ such that $\mathcal{S}pec\,\mathbb{C}\times_{\mathbb{Z}} X = \mathcal{M}_d^{\Theta-st}(Q)$ (Apply Seshadri's integral variant of GIT). And on X we can reduce to finite field k, and get $X(k)$ satisfying

$$|X(k)| = \frac{|R_d(Q)^{st}(k)|}{|PG_d(k)|} = (q-1)\frac{|R_d(Q)^{sst}(k)|}{|G_d(k)|}.$$

By our recursion get $|X(\mathbb{F}_q)| \in \mathbb{Q}(q)$. We have the fact that if the number of rational points of a scheme is counted by a rational polynomial in the size of the field then it is actually a polynomial with integer coefficients: $|X(\mathbb{F}_q)| \in \mathbb{Z}[q]$. And if the scheme is smooth projective it is the Poincare polynomial

$$|X(\mathbb{F}_q)| = \sum_i \dim H^i(\mathcal{M}_d^{st}(Q), \mathbb{Q}) q^{i/2}.$$

In particular $|X(\mathbb{F}_q)| \in \mathbb{N}[q]$.

Here is the final result on the computation of the Poincare polynomial

$$\sum_i \dim H^i(\mathcal{M}_d^{st}(Q), \mathbb{Q}) q^{i/2} = (q-1) \sum_{d=d^1+\cdots+d^s} (-1)^{s-1} q^{-\sum_{k\leq\ell}\langle d^\ell, d^k\rangle} \prod_{k=1}^{s} \prod_{i\in I} \prod_{j=0}^{d_i^k-1} (1-q^{-j})^{-1}.$$

The sum is over all $d = d^1 + \cdots + d^s$ and all $d^s \neq 0$ and $\mu(d^1+\cdots+d^k) > \mu(d)$ for all $k < s$.

This gives all the Betti numbers but not the Euler characteristic (the evaluation of the left hand side at $q = 1$) since there are singularities at $q = 1$ on the right hand side.

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