Toric Geometry
Math 615A - Fall 2011

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Contents

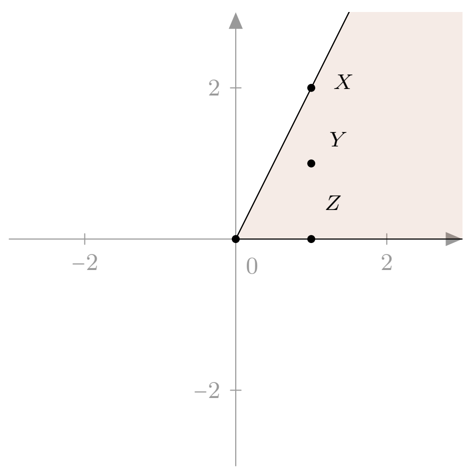
1. Introduction

We have a fixed field k and look at commutative rings over k or often k-algebras. We take a semigroup S and form the semigroup ring $k[S]$ from it in the obvious way. Then S is commutative iff $k[S]$ is and S is a monoid iff $k[S]$ is unital. So we would require S to further be a monoid. Examples include $(\mathbb{Z}_{\geq 0}, +)$ and $(\mathbb{N}, .)$. As a piece of formal notation note that if S is an additive semigroup we can write

$$k[S] = \{\sum_{\text{finite}} a_i \chi^{s_i}\}.$$

So for example if $S = \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ then $\mathcal{Spec}\, k[S] = \mathbb{A}^2_k$. We will use the notation $\chi^{(1,0)} =: x$ and $\chi^{(0,1)} =: y$ to indicate this for concreteness. In general S is a k-basis for $k[S]$ and χ^s are called monomials of $k[S]$

Example 1.1. Take the cone C and let $S = C \cap \mathbb{Z}^2$. Then $k[S] = k[x,y,z]/(xz - y^2)$ as it is generated by the three points u, v and w but the have the relation $u + w = 2v$.



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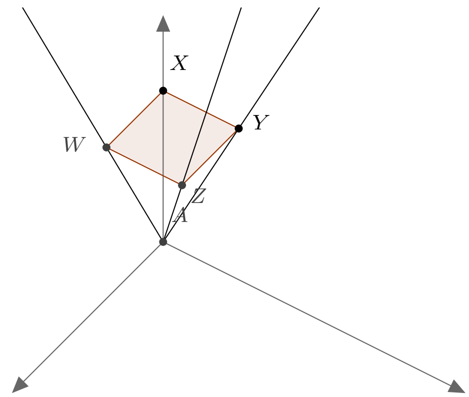
The general procedure for making affine toric varieties is likewise. We take a cone $C \subset \mathbb{R}^n$ and consider $\mathcal{Spec}(k[C \cap \mathbb{Z}^n])$. For the projective case we take a polytope in $P \subset \mathbb{R}^n$ and embed it as $P \times \{1\} \subset \mathbb{R}^{n+1}$. This gives a cone C in $\mathbb{R}^{n+1}$ and we consider $\mathcal{Proj}\, k[C \cap \mathbb{Z}^{n+1}]$. The grading of $C \cap \mathbb{Z}^{n+1}$ comes from the 'height' component in $\mathbb{R}^{n+1}$.

Example 1.2. The square $[0,1] \times [0,1]$ gives $\mathcal{Proj}\, k[x,y,z,w]/(xz - yw) = V(xz - yw) \subset \mathbb{P}^3$. Note that the combinatorial object P can be used to read of the variety in many cases:

for instance since $P = I \times I$ in this case, we get that $\mathcal{P}roj\, k[S] = \mathbb{P}^1 \times \mathbb{P}^1$.



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2. Convex polyhedral cones

What kind of cones are allowed in this construction? These are basically the convex polyhedral cones that are cones we construct from a polytope. The reason we stick to polytope sections is getting finite generation. A circular section does not give finitely many relations.

Recall that a subset $C \subset V \cong \mathbb{R}^n$ is *convex* if whenever $x_1, x_2 \in C$ then $\lambda x_1 + (1-\lambda)x_2 \in C$. C is a *cone* if whenever $x \in C$ then $ax \in C$ for all $a \geq 0$. So C is a *convex cone* if for all $x_1, x_2 \in C$ and $a_1, a_2 \geq 0$ then

$$a_1 x_1 + a_2 x_2 \in C.$$

Remark. If $C \subset \mathbb{R}^n$ is a convex cone, then $C \cap \mathbb{Z}^n$ is a semigroup (monoid).

$$x_1, x_2 \in C \cap \mathbb{Z}^n \Rightarrow x_1 + x_2 \in C \cap \mathbb{Z}^n.$$

For $x_1, \cdots, x_m \in V$ the convex cone generated by $x_1, \cdots, x_m$ is

$$C = \langle x_1, \cdots, x_m \rangle = \{a_1 x_1 + \cdots + a_m x_m : a_i \geq 0\}.$$

Here C is the smallest convex cone containing x.

Definition 1. A convex cone C is polyhedral if it can be generated by a finite set.

Definition 2. A cone C is strongly convex (or pointed) if it does not contain a nonzero linear subspace.

Definition 3. A cone C is simplicial if it can be generated by linearly independent vector.

The picture to keep in mind is that a simplicial cone is generated by a cross section that is a simplex. From now on when we say a 'cone' we mean a convex polyhedral one. Let $C \subset V$ be a cone. Let $y \in V^*$ be a linear function on V.

DEFINITION 4. $y \in V^*$ is a support function of C if $y(x) \geq 0$ for all $x \in C$, written as

$$y|_C \geq 0.$$

Then $H_y = \{y = 0\}$ is a support hyperplane of C. Then $H_y \cap C$ is a face of C.

The set of all face of C is denoted by $[C]$. Faces form a poset under inclusion. A face of C is again a convex polyhedral cone. Note also that if σ, τ are two faces of C then $\sigma \cap \tau$ is also a face. A *facet* is a face of codimension one. A *ray* is a face of dimension one.

LEMMA 1. *Every proper face is contained in a facet.*

3. Dual cone

For a cone $C \subset C$. Let $C^\vee = \{y \in V^* : y|_C \geq 0\}$.

LEMMA 2. *$C^\vee$ is again a convex polyhedral cone and that $(C^\vee)^\vee = C$.*

In general there are tow ways to give a cone. Either by (1) generators: $C = \langle x_1, \cdots, x_m \rangle$ in which case if $y_1, \cdots, y_\ell$ are the normals to the facets then $C^\vee = \langle y_1, \cdots, y_\ell \rangle$. (2) By intersections of half spaces

$$C = \{x : y_1(x) \geq 0, \cdots, y_\ell(x) \geq 0\}$$

(which gives a description of type (1) for the dual cone). Note that it is computationally hard to get from one description to the other one (convex hull is hard to compute)!

LEMMA 3. *$C \subset V$ is strongly convex iff $C^\vee$ has full dimension (by which we mean the span of its elements it the ambient vector space).*

Faces of C and $C^\vee$ are related as follows:

$$d\text{-dimensional faces of } C \leftrightarrow n - d\text{-dimensional faces of } C^\vee$$

via $\tau \mapsto \tau^\perp \cap C^\vee = \tau^*$.

One more piece of notation is that of the polar dual of a polytipe P. If P is put at height 1 in a cone C then the height one polytope of $C^\vee$ is the polar dual of P denoted by P°. In dimension three examples of such dualities are that of the classical Platonic solids.

4. Rational cones

For $C \subset \mathbb{R}^n$, C is called *rational* if we can find generators $x_1, \cdots, x_m \in \mathbb{Q}^n$ for it. This is equivalent to getting m generators in $\mathbb{Z}^n$. If C is strongly convex and rational then it is generated by the first lattice points on the rays.

In general for $M \cong \mathbb{Z}^n$ and $M_\mathbb{R} = M \otimes \mathbb{R} \cong \mathbb{R}^n$, the rational cone $C \subset M_\mathbb{R}$ is generated as

$$C = \langle x_1, \cdots, x_m \rangle$$

for some $x_i \in M$.

LEMMA 4 (Gordon). *If $C \subset \mathbb{R}^n$ is a rational cone then the semigroup $S = C \cap \mathbb{Z}^m$ is finitely generated.*

COROLLARY 1. *$k[S]$ is finitely generated k-algebra and we get $\mathcal{S}pec\, k[S] \subset \mathbb{A}^N$.*

Example 4.1. Take the line passing through $(1, \sqrt{2})$ and create a cone in $\mathbb{R}^2$. Then if $A = k[S]$ is the coordinate ring of the corresponding affine toric variety, the fraction field $\mathrm{ff}(A) = k(x, y)$ implying that we don't have a noetherian ring. ⌟

4.1. Functoriality. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map that maps lattices $\mathbb{Z}^n \to \mathbb{Z}^m$ and cone to cone $C \to D$. Then we get a homomorphism of semigroups $C \cap \mathbb{Z}^n \to D \cap \mathbb{Z}^m$ and thus a ring homomorphism $k[C \cap \mathbb{Z}^n] \to k[D \cap \mathbb{Z}^m]$ and a morphism of corresponding varieties in the contravariant direction.

Notation: $N \cong \mathbb{Z}^n$ and $M = \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$. Let $\sigma \subset N_{\mathbb{R}}$ be a strongly convex rational cone. Then the dual cone is aof full dimension, i.e. $\mathrm{rank}(M)$. Let $S_\sigma = \sigma^\vee \cap M$ the corresponding semigroup and $A = k[S_\sigma]$ the semi-group ring. Then we have the notation

$$U_\sigma = \mathcal{S}pec\, A_\sigma.$$

REMARK. For a cone in dimension 2, the minimal set of generators of the points joining that create

$$\mathrm{Conv}(S_\sigma \setminus \{0\})$$

together with the faces. To find the generators one need computational methods using Groebner basis for the ideal given as follows: Let $\varphi : \mathbb{Z}^k \to M$ be the homomorphism sending generators $e_i \to v_i$ to generators we found in previous step. THen we need to find a set of generators for this $\mathbb{Z}$-module as

LEMMA 5. *A_σ is a domain.*

PROOF. $S_\sigma \subset M$ and

$$A_\sigma = k[S_\sigma \subset k[M] = k[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$$

as a subring. But the latter is a domain. □

LEMMA 6. *If τ is a face of σ then $U_\tau \subset U_\sigma$ as an open set.*

PROOF. In fact we show that U_τ is a principal open set. If τ is defind by $\sigma \cap u^\perp$ then

$$\tau^\vee = \sigma^\vee + (-u.\mathbb{R}).$$

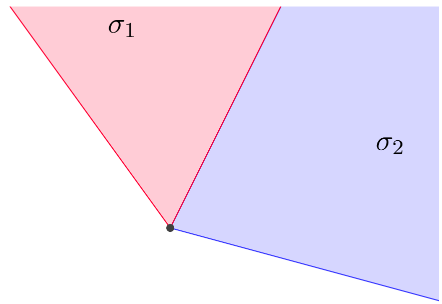
In fact $\supseteq$ is easy and for $\subseteq$ let $v \in \tau^\vee$ then $v|_\tau \geq 0$ and $u|_\sigma \geq 0$ thus for large enough p we have

$$v + p.u \geqslant \sigma.$$

So $v = w - pu \in \sigma^\vee - \mathbb{R}.u$.

And therefore $S_\tau = S_\sigma + (-u)\mathbb{Z}$ hence $A_\tau = A_\sigma[\frac{1}{\chi^u}]$. □

Example 4.2. Given a configuration of cones σ_1 and σ_2 as in



we get two affine opens U_σ and U_{σ_2} that glue along $U_{\sigma_1 \cap \sigma_2}$. ⌟

Definition 5. A fan Δ in $N_{\mathbb{R}}$ is a finite, nonempty set of rational strongly convex cones in $N_{\mathbb{R}}$, such that

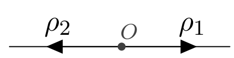
(1) IF $\sigma \in \Delta$ then any $\tau \preccurlyeq \sigma$ is in Δ.

(2) If $\sigma, \tau \in \Delta$ then $\sigma \cap \tau$ is a face of each cone.

Definition 6. The toric variety $X(\Delta)$ associated to the fan Δ is

$$\coprod_{\sigma \in \Delta} U_\sigma / \{U_{\sigma_1}, U_{\sigma_2} \text{ are glues along } U_{\sigma_1 \cap \sigma_2}\}.$$

Example 4.3. The only 1-dimensional toric variety is $\mathbb{P}^1$ contructed from the fan



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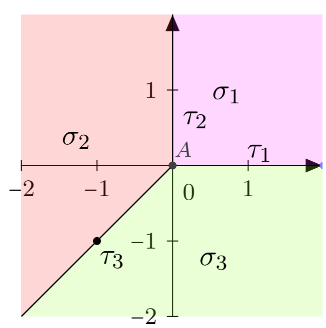
Example 4.4. Here is a 2-dimensional example:



We have

$$\begin{aligned} S_{\sigma_1} &= \mathbb{Z}_{\geq 0}u \oplus \mathbb{Z}_{\geq 0}v, & A_{\sigma_1} &= k[\chi^u, \chi^v], & U_{\sigma_1} &= \mathbb{A}^2 \\ S_{\sigma_2} &= \mathbb{Z}_{\geq 0}(-u) \oplus \mathbb{Z}_{\geq 0}(-u+v), & A_{\sigma_1} &= k[\chi^{-u}, \chi^{-u+v}], & U_{\sigma_2} &= \mathbb{A}^2 \\ S_{\sigma_3} &= \mathbb{Z}_{\geq 0}(-v) \oplus \mathbb{Z}_{\geq 0}(-v+u), & A_{\sigma_1} &= k[\chi^{-v}, \chi^{-v+u}], & U_{\sigma_3} &= \mathbb{A}^2 \end{aligned}$$

giving us $\mathbb{P}^2$. ⌟

Recall that

DEFINITION 7. A morphism (of cones) is

$$\varphi : (N_1, \sigma_1) \to (N_2, \sigma_2)$$

where $\varphi : N_1 \to N_2$ is a homomorphism and $\varphi \otimes \mathbb{R} : N_1 \otimes \mathbb{R} \to N_2 \otimes \mathbb{R}$ maps σ_1 to σ_2.

From such φ we get induced $\psi : U_{\sigma_1} \to U_{\sigma_2}$ called a *toric morphism*.

Example 4.5. Let $N_1 = N_2 = \mathbb{Z}^2$ and $\sigma_1 = \sigma_2 = \mathbb{R}^2_{\geq 0}$. Let $\varphi : \mathbb{Z}^2 \to \mathbb{Z}^2$ be given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in some basis. THen the dual map φ^* is given by transposition of the former matrix. Consequently

$$A_{\sigma_2} \to A_{\sigma_1}$$

is given via $x = \chi^{e_1^*} \mapsto \chi^{e_1^* + e_2^*} = x.y$ and $y = \chi^{e_2^*} \mapsto \chi^{e_2^*} = y$. ⌟

DEFINITION 8. Let $(N_1, \Delta_1) \to (N_2, \Delta_2)$ be fans. A morphism $\Delta_1 \to \Delta_2$ is a group homomorphism $\varphi : N_1 \to N_2$ such that $\varphi_{\mathbb{R}}$ take every cone $\sigma \in \Delta_1$ into some cone of $\tau \in \Delta_2$.

Given (N_1, Δ_1) and (N_2, Δ_2) the product $(N_1 \times N_2, \Delta_1 \times \Delta_2)$ correspond to the product of varieties

$$X(\Delta_1 \times \Delta_2) = X(\Delta_1) \times X(\Delta_2).$$

To check this it suffices to see that

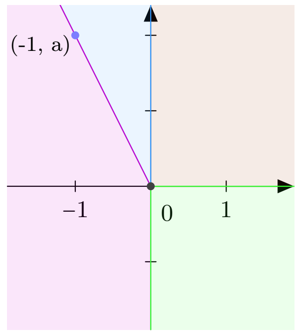
$$U_{\sigma \times \tau} = U_\sigma \times U_\tau,$$

which is the case if $A_{\sigma \times \tau} = A_\sigma \otimes_k A_\tau$ and this also follows if

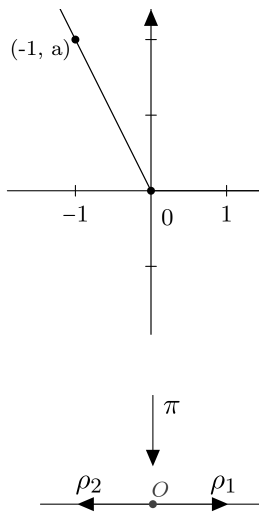
$$S_{\sigma \times \tau} = S_\sigma \times S_\tau$$

as product of semigroups, hence following from our construction.

Example 4.6. Hirzebruch surface F_a.



In fact $F_a = (U_{\sigma_1} \cup U_{\sigma_2}) \cup (U_{\sigma_3} \cup U_{\sigma_4})$ and π in



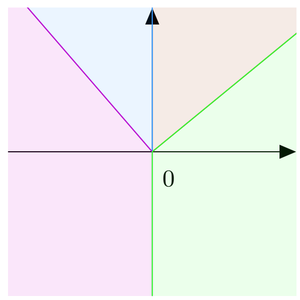
maps the first piece to U_{τ_1} and the second to U_{τ_2}. In fact each open is $\psi^{-1}(U_{\tau_1})$ and $\psi^{-1}(U_{\tau_2})$. One interesting fact is that $F_a \to F_{a+2}$ is a homeomorphism. Let $\mathcal{O}(a) \oplus \mathcal{O}$ be the locally free sheaf on $\mathbb{P}^1$. Let $E \to \mathbb{P}^1$ be the vector bundle of rank 2 it corresponds to. Then we form

$$\mathbb{P}(E) = F_a \to \mathbb{P}^1.$$

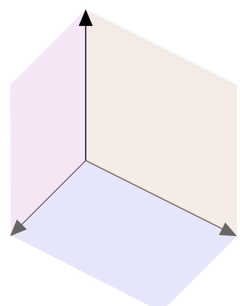
⌟

Remark. F_a and F_{a+2} are not algebraically isomorphic. $F_a \cong F_{-a}$ by toric morphisms.

Another nice toric fact is that any toric variety given by a fan as in the following figure is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ and in fact isomorphic to F_a for some a.



Example 4.7. Let $N = \mathbb{Z}^3$. Let $\sigma = \mathbb{R}^3_{\geq 0}$ and $[\sigma]$ be the set of all faces of σ. Let $\Delta = [\sigma] \setminus \{\sigma\}$.



Let π be the projection from $\mathbb{Z}(1,1,1)$. The image is a $\mathbb{P}^2$. ⌟

Why is $X(\Delta)$ a veriety?

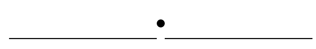
$$A_\sigma = k[S_\sigma]$$

is a domain. Therefore U_σ is a variety (irreducible and reduced) and thus $X(\Delta)$ is irreducible and reduced.

Proposition 1. *$X(\Delta)$ is separated.*

Proof. See Fulton. This follows if one shows that $X(\Delta) \hookrightarrow X(\Delta) \times X(\Delta)$ is closed. This amounts to noting that any two different cones $\sigma, \tau \in \Delta$ can be separated by a hyperplane. $\square$

If we allowed multisets instead of sets, say for instance $\Delta = \{\sigma, \tau, 0\}$ where both σ and τ are rays from the origin in $\mathbb{R}$ heading the positive direction of the x-axis we get the nonseparated toric variety



5. Torus action

Since $0 \preccurlyeq \sigma$ for all σ in the fan (where 0 is the zero cone), $U_0 \subseteq U_\sigma$ and so U_0 is an open subset of $X(\Delta)$. Note that

$$U_0 = \mathcal{S}pec\, A_0 = \mathbb{A}^n \smallsetminus \{x_0 \cdots x_n = 0\} \cong \mathbb{G}_m^n$$

as $A_0 = k[M] = k[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$.

Corollary 2. *$X(\Delta)$ is a rational variety.*

Recall that a rational variety is one that has $k(x_1, \cdots, x_n) \cong k(\mathbb{A}^n)$ as the field of rational functions on it. This is equivalent to the condition that the variety if birational to $\mathbb{A}^n$.

T acts on itself by the following mapping of algebras

$$k[M] \to k[M] \otimes_k k[M], \quad \chi^m \mapsto \chi^m \otimes \chi^m.$$

The same morphism

$$k[M] \to k[M] \otimes_k k[S_\sigma], \quad chi^m \mapsto \chi^m \otimes \chi^m$$

extends the action of T on itself by multiplication. Of course one shall show comptability with inclusions $U_\tau \subset U_\sigma$:

$$\begin{array}{ccc} T \times U_\sigma & \longrightarrow & U_\sigma \\ \cup & & \cup \\ T \times U_\tau & \longrightarrow & U_\tau \end{array}$$

and this is obvious from

$$\begin{tikzcd} k[M] \otimes k[S_\sigma] \arrow[d, hook] & k[S_\sigma] \arrow[l] \arrow[d, hook] \\ k[M] \otimes k[S_\tau] & k[S_\tau] \arrow[l] \end{tikzcd}$$

The k-valued points of $X(\Delta)$ are in one-to-one correspondence with maps $\mathcal{S}pec\, k \to U_\sigma$ or just k-algebra morphisms $k[S_\sigma] \to k$. If R is a k-algebra, then

$$\mathrm{Hom}_{k-alg}(k[S_\sigma], R) = \mathrm{Hom}_{semi-gr}(S_\sigma, (R, .)).$$

The algebra homomorphism is determined by where the monomials go.

Note 5.1. If $R = k[S]$ for a cone S, in general $\mathrm{Hom}_{s-g}(S_\sigma, (R, .))$ is much larger than $\mathrm{Hom}_{s.g}(S_\sigma, S)$. ⌟

So we can rewrite the action on the k-points via

$$\mathrm{Hom}(\mathcal{S}pec\, k, T \times U_\sigma) \to \mathrm{Hom}(\mathcal{S}pec\, k, U_\sigma)$$

via $(\psi, \varphi) \mapsto (\psi.\varphi)$ where $\psi.\varphi : S_\sigma \to k$ is the mapping $m \mapsto \psi(m)\varphi(m)$.

A toric morphism $f : X(\Delta_1) \to X(\Delta_2)$ is equivarient in the following sense: Say f is induced by

$$\varphi : (N_1, \Delta_1) \to (N_2, \Delta_2).$$

Then $T_1 = \mathcal{S}pec\, k[M_1]$ mapsto to $T_2 = \mathcal{S}pec\, k[M_2]$ by restriction of f to U_0's. One can see that this is a group homomorphism. Note that by equivarience we only require $f(t.x) = f(t).f(x)$.

Example 5.2. for $\mathbb{A}^1 \to \mathbb{A}^2$ given by $(0,1) \to (s_1, s_2)$ for $(s_1, s_2) \in \mathbb{Z}^2_{\geq 0}$, we are not mapping the standard tori to each other. ⌟

6. Characters

Let $T = U_0$ given by $(N, 0)$ (where 0 means the zero cone), and $k^\times = V_0$ considered as a toric variety generated by $(\mathbb{Z}, 0)$. The set of characters of T is by definition

$$\mathrm{Hom}_{gr}(N, \mathbb{Z}) = M.$$

An element of this set is $\chi^m : T \to k$ induced by $m \in M$. Here χ^m is a regular function on $T = \mathcal{S}pec\, k[M]$.

Linear representations of T: If T acts on V, we get a representation $T \to \mathrm{GL}(V)$, since T is an abelian group, we have simultaneous digaonalization:

$$V = \oplus_i V_i$$

where V_i is an eigenspace for all $t \in T$. So fixing $v \in V_i$, we have

$$t.v = \lambda_i(t).v$$

and λ_i is a character of T. So we get a decomposition by characters:

$$V = \oplus_m V_m.$$

Example 6.1. If T acts on U_σ, we get an action on $A_\sigma = k[S_\sigma]$ which is an infinite-dimensional k-vector space. So

$$A_\sigma = \oplus_{m \in M} (A_\sigma)_m$$

where

$$(A_\sigma)_m = \begin{cases} k.\chi^m & m \in \sigma^\vee \\ 0 & \text{otherwise.} \end{cases}$$

⌟

Example 6.2. If $\mathcal{F}$ is a T-equivariant linear bundle on U_σ. Then T acts on

$$\mathcal{F}(U_\sigma) = \oplus_m \mathcal{F}(U_\sigma)_m$$

we also get an action of T on the cohomology theory $H^i((X(\Delta), \mathcal{F})$. ⌟

7. 1-parameter subgroups

A one-parameter subgroup is a toric map $k^\times \to T$, i.e. $(\mathbb{Z}, 0) \to (N, 0)$. So the set of all one-parameter subgroups is

$$\mathrm{Hom}_{gr}(\mathbb{Z}, N)N.$$

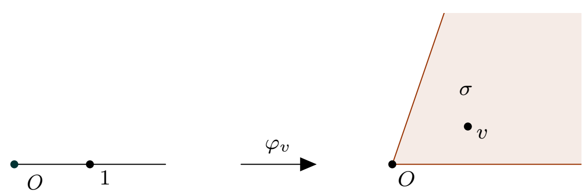
We get a duality pairing $N \times M \to \mathbb{Z}$ in the obvious way.

Lemma 7. *Let $v \in N$ and*

$$k^\times \xrightarrow{\varphi_v} T$$

be a one-parameter subgroup. Let U_σ be an affine toric variety defined by (N, σ). Then φ_v extends to $k \to U_\sigma$ iff $v \in \sigma$.

Proof. If $v \in \sigma$ we get a map of cones



For the other direction given $k^\times \xrightarrow{\varphi_v} T$ defined by $\mathbb{Z} \to N$ via $1 \mapsto v$, we get a dual mapping $v^* : M \to \mathbb{Z}$.

$$\begin{tikzcd}
k[t, t^{-1}] = k[\mathbb{Z}] & k[M] \arrow[l] \\
\cup & \cup \\
k[t] & k[S_\sigma] \arrow[l, dashed]
\end{tikzcd}$$

Image of S_σ lies in $\mathbb{Z}_{\geq 0}$ iff v^* is ≥ 0 on S_σ iff $v \in \sigma$. $\square$

8. Completeness

We recall the valuative criterion first. For a field L a discrete valuation is a ring morphism

$$\nu : K \to \mathbb{Z} \cup \{\infty\}$$

we use the notation for the DVR $R_\nu = \nu^{-1}(\mathbb{Z}_{\geq 0} \cup \{\infty\})$.

THEOREM 8.1 (Valuative Criterion). *If X is a separated variety, X is complete if for any K with discrete valuation, the diagram below always completes (uniquely because of separatedness):*

$$\begin{tikzcd} \mathcal{S}pec\, K \arrow[r] \arrow[d, hook] & X \\ \mathcal{S}pec\, R_\nu \arrow[ur, dashed] & \end{tikzcd}$$

and a map $X \to Y$ is more generally proper if it is separated and

$$\begin{tikzcd} \mathcal{S}pec\, K \arrow[r] \arrow[d, hook] & X \arrow[d] \\ \mathcal{S}pec\, R_\nu \arrow[ur, dashed] \arrow[r] & Y \end{tikzcd}$$

Now let Δ be a fan. The suppose of Δ is

$$|\Delta| = \bigcup_{\sigma \in \Delta} \sigma \subset N_{\mathbb{R}}.$$

The fan Δ is complete if $|\Delta| = N_{\mathbb{R}}$.

THEOREM 8.2. *$X(\Delta)$ is complete iff Δ is complete.*

PROOF. Suppose $X(\Delta)$ is complete but Δ is not. Take $v \in N \smallsetminus |\Delta|$. Then $k^\times \xrightarrow{v} T$ does not extend to $k \to U_\sigma$ for any $\sigma \in \Delta$. Then it does not extend to $k \to X(\Delta)$ so $X(\Delta)$ is not complete. For the other direction if Δ is complete, we check the valuative criterion for field K and discrete valuation ν. Assume $\mathcal{S}pec\, K \to T \subset X(\Delta)$ without loss of generality. We also exploit the assumption that K is a k-algebra. So $\mathcal{S}pec\, K \to X(\Delta)$ gives

$$M \to K^\times \to \mathbb{Z}$$

by some $v \in N$. We choose σ such that it contains v. Then we get a diagram

$$\begin{tikzcd} \mathbb{Z} & K \arrow[l, "\nu"'] & S_\sigma \arrow[l, "s.g"'] \arrow[ll, bend right, "v"'] \arrow[dl, dashed] \\ & R_\nu \arrow[u, hook] & \end{tikzcd}$$

The dotted map gives a ring homomorphism and therefore $\mathcal{S}pec\, R_\nu \to U_\sigma$ as needed. $\square$

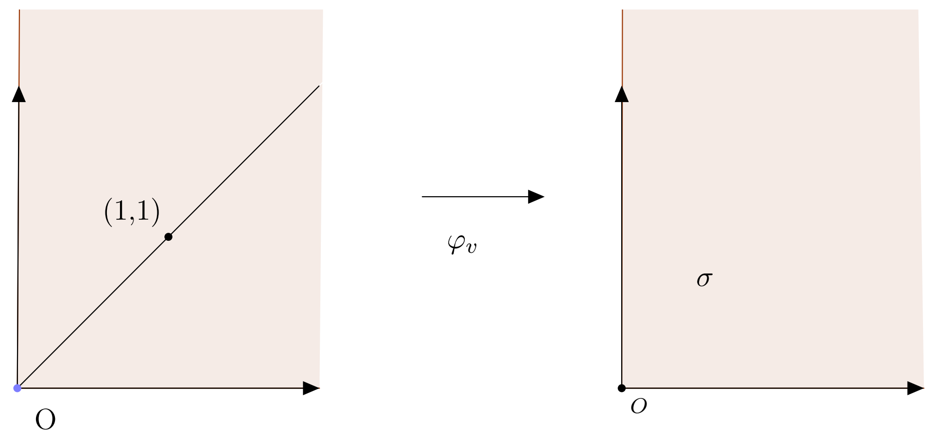
Likewise one can show

THEOREM 8.3. *$X(\Delta_1) \to X(\Delta_2)$ given by $(N_1, \Delta_1) \xrightarrow{\varphi} (N_2, \Delta_2)$ is proper iff $\varphi^{-1}|\Delta_2| = |\Delta_1|$.*

9. Blow-ups

A subdivision of Δ_1 is another cone in the same vector space, such that $|\Delta_2| = |\Delta_1|$ and the identity mapping $\mathrm{id} : \Delta_1 \to \Delta_2$ is a morphism of fans. The morphism induced by id is proper and a birational morphism.

Example 9.1. Here is an example of a subdivision giving the blow-up of $\mathbb{A}^2$ at the origin.



⌟

10. Singularities

Let $\sigma \subset N$ be generated by $\langle v_1, \cdots, v_n \rangle$.

Definition 9. σ is non-singular if $v_1, \cdots, v_m$ can be extended to a basis $v_1, \cdots, v_n$ of N.

For instance if σ is simplicial and $\dim \sigma = n = \mathrm{rk}\, N$, then it is non-singular iff $\det(v_1, \cdots, v_m) = \pm 1$.

Theorem 10.1. *U_σ is nonsingular if and only if σ is non-singular.*

Proof. $\Leftarrow$: let $\sigma = \langle v_1, \cdots, v_m \rangle$ be non-singular. Complete this to $\langle v_1, \cdots, v_n \rangle$, and let their duals be $e_1^*, \cdots, e_n^*$. Then

$$A_\sigma = k[x_1, \cdots, x_m, x_{m+1}^{\pm 1}, \cdots, x_n^{\pm 1}]$$

therefore $U_\sigma = \mathbb{A}^m \times (k^\times)^{n-m}$.

$\Rightarrow$: Assume U_σ is non-singular and $\dim \sigma = m < n = \mathrm{rk}\, N$. Then

$$(\sigma, N) = (\sigma, N_1) \times (0, N_2).$$

From that $U_\sigma = U'_\sigma \times (k^\times)^{n-m}$. So we have that U_σ is non-singular iff U'_σ is nonsingular. So without loss of generality, $\dim \sigma = \operatorname{rk} N$ so then $\sigma^\vee$ is strictly convex. Let $I \subset A_\sigma$ be generated by all χ^m for all $m \in \sigma^\vee$. So $A_\sigma / I \cong k$ so I is a maximal ideal and should correspond to some $x \in U_\sigma$.

If U_σ is nonsingular, then $\dim \mathfrak{m}_x / \dim \mathfrak{m}_x^2 = n$. Note that $\mathfrak{m}_x^2$ is generated by $(\chi^{m_1+m_2})_{m_1,m_2\neq 0}$. But $\mathfrak{m}_x/\mathfrak{m}_x^2$ has a basis consisting of χ^{m_i} over a set of $\{m_i\} = B$ of minimal generators of S_σ. In fact, B contains the first element on each ray. So then the first elements on rays generate S_σ. Therefore $\sigma^\vee$ has n rays, so it is simplicial. That is to say,

$$\sigma^\vee = \langle u_1, \cdots, u_n \rangle$$

with u_i's generating S_σ. So $u_1, \cdots, u_n$ is a basis for M. So $\sigma^\vee$ is nonsingular and consequently σ is nonsingular. $\square$

THEOREM 10.2. *The ring A_σ is integrally closed.*

PROOF. Write

$$\sigma^\vee = \bigcap \text{ half-spaces } H$$

therefore $S_\sigma = \bigcap H \cap M$. So

$$A_\sigma = \bigcap \underbrace{k[H \cap M]}_{\text{integrally closed}} .$$

So A_σ is integrally closed. (In fact A_σ is Cohen-Macauley). $\square$

11. Toric varieties as quotients

So far we have seen that if $X(\Delta)$ is a toric variety, then it is normal, and has a toric T embedded that has an action extending the action of the torus on itself.

If U_σ is smooth then σ is simplicial. The claim is now that if σ is simplicial, then U_σ has finite quotient singularities, i.e. $U_\sigma = \mathbb{A}^n/G$ with G a finite group. Here we are assuming $\operatorname{char} k = 0$. First we see an example:

Example 11.1. Let μ_3 be the group of third roots of unity. This acts on $\mathbb{A}^2$ by

$$\xi(x,y) = (\xi.x, \xi^2.y).$$

Then $\mathbb{A}^2/G = \mathcal{S}pec\, k[x,y]^G$. The induced action on $k[x,y]$ is the obvious one

$$\xi : x \mapsto \xi.x, y \mapsto \xi^2.y.$$

Let $M' \subset M$ be the lattice of invariant monomials as in the picture

$$fig22$$

In fact $x^a y^b$ is fixed by G iff $a + 2b = 0 \mod 3$. Then observe that for

$$S'_\sigma = \sigma^\vee \cap M'$$

we have $\mathbb{A}^2/G = \mathcal{S}pec\, k[S_\sigma]^G = \mathcal{S}pec\, k[S'_\sigma]$. For computational purposes, we consider the exact sequence

$$0 \to M' \to M \to \underbrace{M/M'}_{Q} \to 0$$

of abelian groups, right deriving gives

$$0 \to N \to N' \to \mathrm{Ext}^1(Q, \mathbb{Z}) \to 0$$

so

$$N' = \{(\alpha, \beta); \langle(\alpha, \beta), m'\rangle \in \mathbb{Z}, \forall m' \in M'\} \subset N_{\mathbb{R}}.$$

It follows that if we look at u_1, u_2, u_3 as follows:

$$fig23$$

then N' is realized as

$$N' = \{(\alpha, \beta) : (\alpha, \beta).u_i = 0 \mod 3 (i = 1, 3), (\alpha, \beta).u_2 \in \mathbb{Z} \to \alpha + \beta \in \mathbb{Z}\}.$$

⌟

THEOREM 11.1. *Let σ be a simplicial cone $\sigma = \langle v_1, \cdots, v_n\rangle$. Let $N' \subset N$ be generated as $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n$ then (σ, N') is nonsingular. In fact $U'_\sigma \cong \mathbb{A}^n$. (σ, N) is singular. Let $G = N/N'$ then the assertion is that G act $\mathbb{A}^n$ and $U_\sigma = \mathbb{A}^n/G$.*

PROOF. We have an action of G on $\mathbb{A}^n$ with

$$\chi^{m'} \mapsto e^{2\pi i \langle v, m'\rangle} \chi^{m'}.$$

Now we want to find the invariants. Take $\chi^{m'} \in A'_\sigma$ then

$$\overline{v}.\chi^{m'} = \chi^{m'}, \forall \overline{v} \in G$$

iff $e^{2\pi i, \langle v, m'\rangle} = 1$, if and only if $\langle v, m'\rangle \in \mathbb{Z}$ and consequently $m' \in M$, proving the assertion. □

THEOREM 11.2. *If Δ is simplicial, then $X(\Delta)$ has finite quotient singularities.*

In this case $X(\Delta)$ is an orbifold.

12. Finite toric morphisms

Sublattices of finite index $N' \subset N$ given finite toric morphisms: So A'_σ is a finite A_σ-module. What we proved shows that if U_σ is singular (say simplicial), we can find finite toric morphism $U'_\sigma \to U_\sigma$ where U'_σ is nonsingular.

Here is an example

$$fig24$$

13. Resolution of singularities

Let X be a singular variety. By resolution of singularity we mean $f : Y \to X$, proper birational and such that Y is non-singular. In the toric case, when Δ is a singular fan, we only have to find a subdivision Δ' of Δ such that Δ' is non-singular:

THEOREM 13.1. *Every Δ has a nonsingular subdivision.*

Star subdivision: Let σ be a strongly convex cone, and $\rho \subset \sigma$ a ray. Then $St_\rho\sigma$ is the star subdivision which has maximal cones generate by the ray and a facet of σ, $\langle F, \rho\rangle$ for any facet F.

First barycentric subdivision: Take a rational $\rho_\sigma \subset \mathrm{int}\sigma$ for every cone $\sigma \in \Delta$. And then do iterative start subdivisions to get $bs_1(\Delta)$. So maximal dimension cones correspond to chains of rays we have chosen:

$$\rho_1 \succcurlyeq \rho_2 \succcurlyeq \cdots.$$

Then the observation is that $bs_1(\Delta)$ is simplicial.

DEFINITION 10. Let $\sigma = \langle v_1, \cdots, v_n\rangle$ be of maximal dimension $\mathrm{rk}\, N$ and let

$$N' = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \cdots \mathbb{Z}v_n \subset N.$$

The the multiplicity of σ is

$$\mathrm{mult}(\sigma) = |N : N'|.$$

Another description is the cardinality of the set

$$\{v \in N : v = \sum \alpha_i v_i, 0 \le \alpha_i < 1\}$$

for obvious reasons. And it is also immediate that σ is nonsingular if and only if its multiplicity is one.

Let σ be simplicial and singular and $0 \neq v = \sum \alpha_i v_i$ an element in the above set. Now star subdivide at $\langle v\rangle$. Then the multiplicity of each maximal cone

$$\sigma_i = \langle v_1, \cdots, \widehat{v_i}, \vdots, v_n, v\rangle$$

is of multiplicity strictly less that multiplicity of σ:

$$\mathrm{mult}\sigma_i = |\det(v_1, \cdots, \widehat{v_i}, \cdots, v_n, v)| = |\alpha_i \det(v_1, \cdots, v_i, \cdots, v_n)| = \alpha_i \mathrm{mult}\sigma.$$

REMARK. If σ is nonsingular, then for $\rho = \langle v_1 + \cdots + v_n\rangle$, $St_\rho\sigma$ is the blow-up at the local $0 \in \mathbb{A}^n$, and if $\rho = \langle v_1 + \cdots + v_m\rangle$ the it is the blow-up of

$$\mathbb{A}^{n-m} \subset \mathbb{A}^n.$$

By this algorithm we have a resolution which is not canonical in any sense. Is there a minimal resolution then? Recall that the minimal resolution is unique up to a sequence of flip-flops.

$$fig25$$

It is however easy to see that if X is a toric surface then it has a unique minimal (toric) resolution.

$$fig26$$

In general if a (non-toric) surface has a G-action, the G action lifts to its resolutions. This will show that the minimal resolution of a toric surface is unique.

14. Resolution graph and exceptional divisors

The resolution graph has vertices $E_1, \cdots, E_n$ for the exceptional divisors, and the edges are the intersection points as in the following picture:

$$fig27$$

The self-intersection numbers of the exceptional divisors can be read off in the toric case via

$$v_{i-1} + v_{i+1} = -a_i v_i$$

where v_i's are the generators of E_i's.

Example 14.1 (ADE-singularities)**.** All self-intersection numbers are -2.

$$fig27$$

⌟

This picture generalizes as far as finding the resolution graph of the exceptional divisors

If σ is a singular 2-dimensional cone up to isomorphism. It has the form

$$fig28$$

up to isomorphism, where $k < m$. For this just add or subtract enough $(m, \ell \pm m)$ by lattice isomorphisms

$$fig29$$

And then the resolution data can be found from this canonical cones:

$$fig30$$

for instance $\text{mult}\sigma = |\det \begin{pmatrix} 0 & m \\ 1 & -k \end{pmatrix}| = m$ and $\text{mult}\sigma_2 = |\det \begin{pmatrix} 1 & m \\ 0 & -k \end{pmatrix}| = k$.

Example 14.2. Hirzebruch Yung continuous fractions

$$E_i.E_i = -a_i$$

then

$$\frac{m}{k} = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \cdots}}.$$

⌟

15. T-orbits

For the action of T on U_σ, the claim is that orbits in U_σ are in one-to-one correspondence with faces of σ.

If $\rho \preccurlyeq \tau$ then $U_\rho \subset U_\tau$ and U_τ contains smaller orbits: O_τ is smaller than O_ρ. Here is the definition of orbit $O_\tau \subset U_\sigma$.

Let $N \cong \mathbb{Z}^n$ and suppose $\dim \tau = m$. If $m = n$ let $I \subset A_\sigma$ be the ideal generated by χ^u for all $u \neq 0$, this cuts $O_\tau = V(I)$ which is a point in this case. If $\dim \tau \leq n$ then Let

$$M(\tau) = M \cap \tau$$

then $I \subset A_\tau$ will be the ideal generated by all $(\chi^u)_{u \notin \tau^\perp}$ in which case

$$\mathcal{S}pec\, A_\tau / I = O_\tau = (k^\times)^{\operatorname{rk} M(\tau)} = (k^\times)^{n-m}.$$

What are the points in O_τ? If $x \in U_\tau$ correspond to the semigroup homomorphism $x : S_\tau \to (k, .)$, then $x \in O_\tau$ if and only if $x : u \mapsto 0$ if $u \notin \tau^\perp$. Giving $x \in O_\tau$ is hence the same data as a semigroup morphism $M(\tau) \to (k, .)$.

Claim: O_τ is a T-orbit. Fix a special point $x_\tau \in O_\tau$. Then $T.x_\tau = O_\tau$.

Lemma 8. $U_\sigma = \sqcup_{\tau \preccurlyeq \sigma} O_\tau$.

Proof. Show that any $x \in U_\sigma$ lies in a unique $O_\tau, \tau \preccurlyeq \sigma$. Say $x : S_\sigma \to k$ is a semigorup morphism. Then $x^{-1}(0) \subset S_\sigma$ is a prime deal in S_σ. Then by the homework problem, $x^{-1}(0) = S_\sigma \smallsetminus \{u \in \tau^\perp\}$ for some face $\tau \preccurlyeq \sigma$. We extend this to S_τ and this completes the proof. □

Example 15.1.

$$fig31$$

⌟

Let $V(\tau) = \overline{O}_\tau$ be the closure of O_τ in $X(\Delta)$. Check that

$$V(\tau) = \sqcup T - \text{orbits}.$$

Example 15.2.

$$fig32$$

$V(\rho_i) \cap V(\rho_{i+1})$ is a point and $V(\rho_1) \cap V(\rho_3)$ is empty. ⌟

The claim is that $V(\tau)$ is itself a toric variety of dimension $n - \dim \tau$. To prove this one needs to observe that $V_\tau \subset U_\sigma$ is $V(J)$ for the ideal generated by

$$fig33$$

Note 15.3. $V(\tau) \hookrightarrow X(\Delta)$ is not a toric morphism. ⌟

Morphisms:

$$\rho : (N_1, \Delta_1) \to (N_2, \Delta_2)$$

f is equivariant, i.e. it takes orbits to orbits, $O_\tau \mapsto O_\sigma$. σ is the smallest cone containing $\varphi(\tau)$. If $\varphi : N_1 \to N_2$ is surjective or has a finite cokernel (i.e. $\varphi_{\mathbb{R}} : N_1, \mathbb{R} \to N_2, \mathbb{R}$ is surjective) then $f : T_1 \to T_2$ is surjective and orbits maps onto orbits.

16. Homological properties

We know that the Euler characteristic satisfies the science relation and trivial fibration axioms:

(1) $\chi(X) = \chi(U) + \chi(X \setminus U)$ for open $U \subset X$.

(2) $\chi(X \times Y) = \chi(X)\chi(Y)$.

So we can find the Euler characteristic of $X(\Delta)$ of a fan Δ from its toric orbits:

$$X(\Delta) = \coprod_{\sigma \in \Delta} O_\sigma.$$

So

$$\chi(X(\Delta)) = \sum_\sigma \chi(O_\sigma).$$

But we know that $O_\sigma = (\mathbb{C}^*)^{n - \dim \sigma}$. Therefore

$$\chi(O_\sigma) = \begin{cases} 0 & \dim \sigma < n \\ 1 & \dim \sigma = n \end{cases}.$$

We conclude that

$$\chi(X(\Delta)) = \text{ number of } n\text{-dimensional cones in } \Delta \; = \; \text{number of } T\text{-fixed points }.$$

For further topological information one should study the Mixed Hodge theory of these varieties. When X is nonsingular and is stratified into locally closed subsets, then one can recover the cohomology ring $H^*(X, \mathbb{C})$ from the cohomology of the strata.

CHAPTER 1

Divisors and line bundles

1. Divisors

1.1. Review. Let X be a normal variety of dimension n. A divisor on X is a formal finite sum

$$D = \sum a_i D_i$$

where $a_i \in \mathbb{Z}$ and D_i are irreducible $(n-1)$-dimensional subvarieties. $\mathrm{Div}(X)$ is the set of divisors on X. This is an abelian group. We call D effective if all $a_i \geq 0$ and write this as $D \geq 0$.

For any $f \in k(X)^*$ we have $\mathrm{div}(f) = \sum a_i D_i$ as a finite sum of divisors. In that case it is called a principal divisor. Here

$$a_i = \mathrm{ord}_{D_i} f$$

which is positive if f vanishes along D_i with multiplicity a_i and is negative if f has a pole of order a_i. We have $\mathrm{div}(f.g) = \mathrm{div} f + \mathrm{div} g$ giving a homomorphism from the multiplicative group to the additive group:

$$k(X)^\times \to \mathrm{Div}(X).$$

If for a covering U_i we have $D|_{U_i} = \mathrm{div}(f_i)$ for some $f_i \in k(U_i)^\times = k(X)^\times$ we call D a Cartier (or locally principal) divisor. The subgroup of these divisors is denoted by $\mathrm{CDiv}(X) \subset \mathrm{Div}(X)$.

We can recover the codimension 1 Chow group from this:

$$A_{n-1} = \mathrm{Div}(X)/\{\text{ principal divisors }\} = \mathrm{Div}(X)/\text{ linear equivalence}$$

The linear equivalence reads $D \sim E$ if $D - E = \mathrm{div}(f)$.

Recall the sheaves $\mathcal{O}_X(D)$ for a divisor D:

$$\mathcal{O}_X(D)(U) = \{f \in k(X) : D + \mathrm{div}(f) \geq 0\}$$

which is the set of $f \in k(X)$ such that f can have a pole of order a_i along D_i if $a_i \geq 0$ and f must vanish at least to order a_i along D_i if $a_i < 0$. For instance $\mathcal{O}(-D_i) = I_{D_i}$ for an irreducible component D_i, i.e. the ideal sheaf of D_i. And $\mathcal{O}(D_i)$ are the functions that have at most one pole along D_i.

If D is Cartier, $\mathcal{O}(D)$ is invertible. In fact, in that case on each U_i we have

$$\mathcal{O}(D)|_{U_i} = \frac{1}{f_i}\mathcal{O}_{U_i} \subset k(X).$$

We have

$$\mathrm{Pic}(X) = \mathrm{CDiv}(X)/ \text{ linear equivalence}$$

which is the group of isomorphism classes of invertible sheaves, with the operation given by tensor product of sheaves. One important property to recall is that

Proposition 2. *$\mathcal{O}_X(D) \cong \mathcal{O}_X(E)$ if and only if $D \sim E$.*

If X is smooth, then $\mathrm{CDiv}(X) = \mathrm{Div}(X)$. If $\mathrm{CDiv}(X) = \mathrm{Div}(X)$, then X is called fractional. If R is a UFD then every divisor on X is principal.

1.2. Toric divisors. If Δ is a fan, and $\rho_1, \cdots, \rho_d$ are rays of Δ. Let D_i be the irreducible divisor that is induced by ρ_i.

Let

$$\mathrm{Div}_T(X(\Delta)) = \{\sum_{i=1}^{d} a_i D_i : a_i \in \mathbb{Z}\} = \mathbb{Z}^d.$$

Lemma 9. *For any $D \in \mathrm{Div}(X(\Delta))$ there exists $E \in \mathrm{Div}_T(X(\Delta))$ such that $D \sim E$.*

Proof. $T \subset X(\Delta)$ is just $\mathcal{S}pec\, k[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$ which is a UFD. Then $D|_T = \mathrm{div}(f)$. So let $E = D - \mathrm{div}(f)$. E has no component on T therefore

$$E = \sum a_i D_i.$$

□

Corollary 3.

$$A_{n-1}(X(\Delta)) = \mathrm{Div}/\sim = \mathrm{Div}_T(X(\Delta))/\sim.$$

Lemma 10. *For any $m \in M$,*

$$\mathrm{div}(\chi^m) = \sum a_i D_i$$

where $a_i = \langle m, v_i \rangle$.

Proof. Compute $\mathrm{ord}_{D_i}\chi^m$. Do this in U_{ρ_i}.

$$A_{\rho_i} = k[x_1, x_2^{\pm 1}, \cdots, x_n^{\pm 1}].$$

Then $U_{\rho_i} = \mathbb{A}^1 \times (k^\times)^{n-1}$. So $D_i = \{0\} \times (k^\times)^{n-1}$. Therefore $\mathrm{ord}_{D_i} x_1 = 1$. □

Lemma 11.

$$\mathrm{Div}_T(X(\Delta)) \cap \{\textit{principal divisors}\} = \{\mathrm{div}(\chi^m) : m \in M\}.$$

PROOF. $D = \mathrm{div}(f) = \sum a_i D_i$ such that $D|_T = 0$, then f in invertible on T. But
$$k[x_1^{\pm 1}, \cdots, x_n^{\pm 1}]^\times = \{a.\chi^m : a \in k^\times, m \in M\}.$$
Hence $\mathrm{div}(f) = \mathrm{div}(a\chi^m) = \mathrm{div}(\chi^m)$. □

COROLLARY 4. $A_{n-1}(X(\Delta)) = \mathrm{Div}_T(X(\Delta))/M$.

Example 1.1. For $\mathbb{P}^2$, $\mathrm{Div}_T\mathbb{P}^2 \cong \mathbb{Z}^3$ and
$$e_1 \mapsto D_1 - D_3, \quad e_2 \mapsto D_2 - D_3.$$
Hence $\mathrm{Pic}(\mathbb{P}^2) = A_1(\mathbb{P}^2) = \mathbb{Z}$. ⌟

Example 1.2. In
$$fig34$$
$\mathrm{Div}_T U_\sigma = \mathbb{Z}^2$ and $M \mapsto \mathrm{Div}_T U_\sigma$ via $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$. So
$$A_1(U_\sigma) = \mathbb{Z}/2\mathbb{Z}.$$
In fact in this case
$$\{\text{principal divisors}\} = \mathrm{CDiv}(X) \subsetneq \mathrm{Pic}(X).$$
⌟

2. Cartier T-divisors

If $D = \sum D(\rho_i)$ (locally) we call it a Cartier T-divisor.

LEMMA 12. *Every T-Cartier divisor on U_σ is principal.*

PROOF. $D \in \mathrm{Div}_T(U_\sigma)$. Assume D is Cartier. The invertible sheaf $\mathcal{O}(D)$ is coherent so let
$$P = \Gamma(U_\sigma, \mathcal{O}(D))$$
be an A_σ-module. We need to prove that P is a free A_σ-module. Then $P \cong A_\sigma$ and $\mathcal{O}(D)$ is trivial. Then D is principal.

Now T acts on P and we have a grading $P = \oplus_m P_m$. Assume σ is full-dimensional. For the special point $x_\sigma \in U_\sigma$ we have $P/\mathfrak{m}_{x_\sigma} = k$. Choose $f \in P_m$ such that $\overline{f} \neq 0$. Then f is a generator of $O(D)$ near x_σ. Then by the action of T we have $f = c.\chi^m$ for a character χ^m. Near x_σ, D is defined by $\frac{1}{f} = c\chi^{-m}$. Thus χ^{-m} defined D everywhere. □

Let $D \in \mathrm{CDiv}_T(U_\sigma)$. The above theorem shows that $D = \mathrm{div}\chi^u$ for some $u \in M$. We observe that if
$$\mathrm{div}\chi^{u_1} = \mathrm{div}\chi^{u_2}$$
then $\frac{\chi^{u_1}}{\chi^{u_2}} = \chi^{u_1 - u_2}$ which is invertible on U_σ. Then $u_1 - u_2 \in \sigma^\perp$.

For any $u \in M$ we can view u as a linear functional on σ. u is integral if $u(v) \in \mathbb{Z}$ for any $v \in N \cap \sigma$. Then

$$\mathrm{CDiv}_T(U_\sigma) = \{ \text{ integral linear functions on } \sigma \ \}.$$

For the $\supseteq$ direction if $u(\sigma) : \sigma \to \mathbb{R}$ maps $N \cap \sigma$ to $\mathbb{Z}$ then extend this to a linear function:

$$u(\sigma) \in M_{\mathbb{R}}$$

and also conclude that $u(\sigma) \in M$.

If $D \in \mathrm{CDiv}_T X(\Delta)$ then for every $\sigma \in \Delta$ $u_(\sigma)$ is an integral linear function on σ:

$$D|_{U_\sigma} = \mathrm{div}\chi^{u(\sigma)}.$$

If $\tau \le \sigma$ then $u(\sigma)|_\tau = u(\tau)$. So $\{u(\sigma)\}_{\sigma \in \Delta}$ gives a piecewise linear (linear on every cone) integral (i.e. taking Δ to $\mathbb{Z}$) on Δ. So this classifies all Cartier T-divisors on $X(\Delta)$:

$$\mathrm{CDiv}_T X(\Delta) = \{\varphi_D : \text{piece-wise linear integral functions on } \Delta\}.$$

So the Picard group is realized as the quotient

$$\{ \text{ piecewise linear integral functions } \}/\{ \text{ global linear integral functions } \}.$$

This is becuase we have

$$\begin{tikzcd}
\mathrm{CDiv}_T(X(\Delta)) = \pi^{-1}\mathrm{Pic}(X) \arrow[r, hook] \arrow[d, two heads] & \mathrm{Div}_T(X) \arrow[d, two heads, "\pi"] \\
\mathrm{Pic}(X) \arrow[r, hook] & A_{n-1}(X)
\end{tikzcd}$$

Example 2.1. For $X = \mathbb{P}^1$ we have $\mathrm{CDiv}_T(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}$ and $\mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z}$ easily! ⌟

LEMMA 13. $\mathrm{Pic}(X(\Delta))$ *is a finitely generated free group. For this we need a completeness assumption or at least that Δ has a cone of full dimension.*

PROOF. We show that $\mathrm{Pic}X(\Delta)$ has no torsion: If $[\varphi_D] \in \mathrm{Pic}X(\Delta)$ satisfies

$$[\ell\varphi_D] = 0$$

then $\ell.\varphi_D$ is globally linear, so φ_D is globally linear. Thus $D \sim 0$. □

Example 2.2.

$$\mathrm{Pic}U_\sigma = 0$$

and $A_1 U_\sigma = \mathbb{Z}/2\mathbb{Z}$. Here D_1 is not Cartier and

$$\varphi_D(v) = \frac{1}{2}$$

so it is not integral.

$$fig40$$

⌟

Example 2.3. Here D_1 is a Cartier divisor but $\mathbb{Z}/2\mathbb{Z} = \mathrm{Pic}(X)$ as we do not have the full-dimensionality condition satisfied.

$$fig41$$

⌟

If σ is simplicial, for any $D \in \mathrm{Div}_T(U_\sigma)$ there is $\ell > 0$ such that $\ell.D$ is Cartier. In this case we say D is $\mathbb{Q}$-Cartier. If σ is not simplicial however, for any choice of numbers $a_1, \cdots, a_k$ over the rays, a linear functional φ_D has to satisfy some extra relations to be $\mathbb{Q}$-Cartier. This motivated the following notion and theorem:

Definition 11. X is $\mathbb{Q}$-fractional if for any $D \in \mathrm{Div}(X)$, we have integer $\ell > 0$ such that $\ell D \in \mathrm{CDiv}(X)$.

Theorem 2.1. *Δ is simplicial if and only if $X(\Delta)$ is $\mathbb{Q}$-fractional.*

Notation: In [F] the notation for $u(\sigma)$ is actually $-u(\sigma)$ and $\psi_D = -\varphi_D$. So for ψ_D a piecewise linear function, $D = \sum a_i D_i$, with $a_i = -\psi_D(v_i)$ if the corresponding divisor and if $\psi_D|_\sigma = u(\sigma)|_\sigma$ then

$$D = \mathrm{div}\chi^{-u(\sigma)}$$

and the linear bundle is trivialized via

$$\mathcal{L}|_{u_\sigma} = \chi^{u(t)}\mathcal{O}_X|_{u_\sigma}.$$

Example 2.4. Let $D = -2D_1 + D_2 - D_3$ in the following picture we have all $u(\sigma_i)$'s and all $\mathcal{L}(U_{\sigma_i}) = \chi^{u(\sigma_i)}A_{\sigma_i}$'s illustrated.

$$fig46$$

⌟

The global sections $\Gamma(X(\Delta), \mathcal{L})$ can be read off from such pictures: Let

$$P_D = \bigcap_{\sigma\in\Delta}(u(\sigma) + \sigma^\vee).$$

If $u \in P_D \cap M$, then χ^u is a rational function on $X(\Delta)$ so that χ^u is a section of $\mathcal{L}$ on each U_σ:

$$\chi^u \in \Gamma(X, \mathcal{L}).$$

So $\{\chi^m : m \in P_D \cap M\}$ is a basis for $\Gamma(X(\Delta), \mathcal{L})$.

Remark. If D is T-invariant (in particular true for all T-divisors), $\mathcal{L} = \mathcal{O}(D)$ is T-equivariant. Also $\mathcal{L}(U_\sigma)$ has an induced T action so the space of global sections $\Gamma(X(\Delta), \mathcal{L})$ has a T-action according to which we have a grading

$$\Gamma(X(\Delta), \mathcal{L}) = \bigoplus_{m\in M}\Gamma(X(\Delta), \mathcal{L})_m.$$

Example 2.5. $\mathbb{F}_1$ embeds in $\mathbb{P}^4$ by $D = D_1 + D_2 + D_3$

$$fig47$$

⌟

REMARK. If D is effective then $1 \in \mathcal{O}(D)(X)$ reconstructs D via $\mathrm{div}(1) = D$.

Note that when Δ is complete, P_D is compact. This gives a proof of finite-dimensionality of the space of global sections. In general P_D is described by the condition

$$u \in P_D \text{ if and only if } u \geq \psi_D.$$

In general the conditions are

$$u \geq \psi_D \text{ on every } v_i \text{ i.e. } u(v_i) \geq -a_i.$$

So

$$P_D = \{u \in M_{\mathbb{R}} : \langle u, v_i \rangle \geq -a_i\}.$$

The relations in linear equivalence translate to

$$P_{D+\mathrm{div}\chi^m} = P_D + m$$
$$P_{\ell.D} = \ell.P_D$$
$$P_{D+E} = P_D + P_E$$

where in right-hand side of the last relation is the Miskowski sum and in general one needs non-emptiness condition on all polytopes above.

3. Maps to projective spaces

Recall that $\mathcal{L}$ is said to be globally generated if for any $x \in X$, there is a section $s \in \Gamma(X, \mathcal{L})$ such that $s(x) \neq 0$. If $\mathcal{L}$ is globally generated and $s_0, \cdots, s_n$ is a basis for $\Gamma(X, \mathcal{L})$ we get a well-defined map to $\mathbb{P}^n$.

THEOREM 3.1. *If Δ is complete, $\mathcal{O}(D)$ is globally generated if and only if ψ_D is convex.*

PROOF. If $\mathcal{O}(D)$ is globally generated, then $u(\sigma)$ is a global section for every cone. So $u(\sigma) \geq \psi_D$ for all σ, therefore ψ_D is convex. Conversely if ψ_D is convex, then $u(\sigma) \geq \psi_D$ for all maximal dimensional σ. So there is a section $s \in \Gamma(X(\Delta), \mathcal{L})$ such that $s(x_\sigma) \neq 0$. So

$$V(\text{ global sections of } \mathcal{L} \text{ }) = \{x \in X(\Delta); \mathcal{L} \text{ is not globally generated }\}$$

which is a closed set and T-invariant so it is a union of T-orbits containing some x_σ. This set is φ. □

one section missi

4. Ample bundles

Let X be a complete variety and $\mathcal{O}(D)$ an invertible sheaf on X. The section ring of $\mathcal{O}(D)$

$$R(\mathcal{O}(D)) = \oplus_{m=0}^{\infty}\Gamma(X, \mathcal{O}(mD))$$

is a graded ring. When $\mathcal{O}(D)$ is ample then $R(\mathcal{O}(D))$ is finitely generated and $X = \mathcal{P}roj(R(\mathcal{O}(D)))$. Note that it is not in general the case that $R(\mathcal{O}(D))$ is finitely generated. It is in fact a question of the minimal model program to investigate whether $R(\mathcal{O}(D))$ is finitely generated when $D = K_X$.

In the toric case the situation is much better. Given D we have $\mathcal{O}(D)$ and P_D as before and then $\Gamma(X(\Delta), \mathcal{O}(mD))$ has a basis $M \cap mP_D$.. Then $R(\mathcal{O}(D)) = k[C_D \cap M \times \mathbb{Z}]$ which is always finitely generated, normal, and also Cohen-Macauly.

We already know that if $\mathcal{O}(D)$ is ample then we recover $X(\Delta)$ from P_D via

$$X(\Delta) = \mathcal{P}roj\, R(\mathcal{O}(D)).$$

We now show that we can recover Δ and ψ_D form P_D when D is ample.

$$fig50$$

Claim: $\partial C_D^{\vee}$ is the graph of a piecewise linear function on $\mathbb{N}_{\mathbb{R}}$. In fact it is the graph of $-\psi_D$. Also Δ is the image of the fan ∂C_D under projection from $(0,1)$.

Proof. Start with Δ and ψ_D strictly convex and construct

$$C_D^{\vee} = \{(v,z) : z \geq -\psi_D(v)\}.$$

Then check that $(C_D^{\vee})^{\vee}$ is the cone over P_D. For this note that

$$(C_D^{\vee})^{\vee} = \{(u,y) : (u,y).(v,z) \geq 0, \forall (v,z) \in C_D^{\vee}\}$$

and hence

$$(C_D^{\vee})^{\vee} \cap \{y = 1\} = \{(u,a) : u.v + z \geq 0\}.$$

□

So the general picture is that

$$\begin{array}{ccc}
\underbrace{\{\text{faces of } C_D^{\vee}\}}_{F, \dim d} & \longleftrightarrow & \underbrace{\{\text{ faces of } C_D \}}_{F^*.\dim n+1-d} \\
\updownarrow & & \updownarrow \\
\underbrace{\{\text{ cones of } \Delta \}}_{\pi(F)=\sigma, \dim d} & & \underbrace{\{\text{ faces of } P_D \}}_{P_\sigma = F^* \cap \{y=1\}, \dim = n-d}
\end{array}$$

then $\sigma \perp P_\sigma$ and Span$F \perp$ SpanF^*. P_D is a lattice polytope with vertices being the lattice points $u_(\sigma)$ for maximal cones σ. If D is very ample, then P_D induces an embedding $X \hookrightarrow \mathbb{P}^{\#\{M \cap P_D\}-1}$.

Example 4.1. Here is the embedding $\mathbb{F}_1 \hookrightarrow \mathbb{P}^4$ given via $x \mapsto (\chi^{u_1}(x) : \cdots : \chi^{u_5}(x)) = (x_1 : \cdots : x_5)$ then since $u_1 + u_5 = u_2 + u_4$ the embedding is cut out by

$$x_1 x_5 = x_2 x_4, \quad x_3 x_4 = x_2 x_5, \quad x_1 x_3 = x_2^2.$$

$$fig51$$

⌟

Start with any lattice polytope $P \subset M_\mathbb{R}$ and apply the construction to get ψ_D, a piecewise-linear function, and Δ, the projection of ψ_D. Thus from P we get a polarized toric variety $(X(\Delta), \mathcal{O}(D))$. Note that if we start from (Δ, ψ_D) with ψ_D convex, and get P_D and then go backwards to some (Δ', ψ'_D) we get a strictly convex function ψ'_D and Δ is a subdivision of Δ'.

Example 4.2. Here let $D = D_4$ then $\mathcal{O}(D_4)$ is globally generated but not ample.

$$fig52$$

Note that $\mathcal{O}(D)$ is just the pull back of $\mathcal{O}(D')$ which is ample. One thing we have not mentioned before though is that to pull back a bundle we may just pull-back the piecewise linear function it corresponds to.

$$\begin{tikzcd} X(\Delta) \arrow[d, "f"'] \arrow[dr] & \\ X(\Delta') \arrow[r] & \mathbb{P}^2 \end{tikzcd}$$

⌟

If ψ_D is a convex function on Δ, then there is a unique fan Δ', and a strictly convex $\psi_{D'}$ on Δ' such that $f : \Delta \to \Delta'$ satisfies $\psi_D = f^* \psi_{D'}$.

5. Cox ring

When X is a complete variety, the Cox ring of X is

$$\text{Cox}(X) = \oplus_{\mathcal{O}(D) \in \text{Pic}(X)} \Gamma(X, \mathcal{O}(D))$$

which is graded by the Pic(X) (in a meaningful way if the Picard group is a lattice):

$$\Gamma(X, \mathcal{O}(D)) \times \Gamma(X, \mathcal{O}(E)) \to \Gamma(X, \mathcal{O}(D + E)).$$

In the toric case, when Δ is a complete, nonsingular fan with rays $\rho_1, \cdots, \rho_d$ we have a projection

$$0 \to K \to \mathbb{Z}^d \xrightarrow{e_i \mapsto v_i} N \to 0$$

which dualizes to

$$0 \to M \to \mathbb{Z}^d \xrightarrow{\pi} \mathrm{Pic}(X(\Delta)) \to 0.$$

Note that $\mathbb{Z}^d = \mathrm{Div}(X(\Delta))$ having the cone of effective divisors inside it. So π restricted to the effective locus, can be thought of as the projection from a real cone in $\mathbb{R}^d$ to $\mathrm{Pic}(X(\Delta)) \otimes \mathbb{R}$.

Proposition 3. $\pi^{-1}(\mathcal{O}(D)) \cap \mathbb{R}^d_{\geq 0} = P_D$.

Proof.

$$\pi^{-1}\mathcal{O}(D) \cap \mathbb{Z}^d_{\geq 0} = \{D' : D' \text{ is effective }, D' \sim D\}.$$

Assume D is effective then $\pi^{-1}(\mathcal{O}(D)) - D \subset M$ is then

$$\{D' - D : D' \text{ is effective }, D' - D \neq 0\} = \{m \in M \cap P_D\}.$$

$\square$

Corollary 5. *In the above case*

$$\mathrm{Cox}(X(\Delta)) = k[\mathbb{Z}^d_{\geq 0}] = k[X_1, \cdots, X_d].$$

CHAPTER 2

Cohomology

We use our basis toric affine patching for $X(\Delta)$ to compute $H^*(X,\mathcal{F})$ for the line bundle $\mathcal{F} = \mathcal{O}(D)$.

Example 0.1. Let us choose $-1, 0, -1$ for the coefficients of ρ_1, ρ_2 and ρ_3 is the usual picture of $\mathbb{P}^2$. Then $D = D_1 + D_3$ and we have $u(\sigma_1) = (-1,0), u(\sigma_2) = (1,0), u(\sigma_3) = (-1,2)$ giving our piecewise-linear function. Then it is easy to form the regions of the 2-space corresponding to $\mathcal{F}(U_{\sigma_i})$ and $\mathcal{F}(U_{\rho_j})$'s. So we have

$$0 \to \mathcal{F}(U_{\sigma_1}) \times \mathcal{F}(U_{\sigma_2}) \times \mathcal{F}(U_{\sigma_3}) \to \mathcal{F}(U_{\rho_1}) \times \mathcal{F}(U_{\rho_2}) \times \mathcal{F}(U_{\rho_3}) \to \mathcal{F}(U_0) \to 0.$$

and since we have

$$H^i(X(\Delta), \mathcal{O}(D)) = \bigoplus_{u \in M} H^i(X(\Delta), \mathcal{O}(D))_u$$

we form this sequence for any u. For instance $u = (0,-1)$ gives

$$0 \to k \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} k^2 \xrightarrow{\begin{pmatrix}1 & 1\end{pmatrix}} k \to 0.$$

⌟

Fix $u \in M$ and let $|\Delta|$ be the support of Δ. Let

$$Z(u) = \{v \in |\Delta| : u(v) \geq \psi_D(v)\} \subset |\Delta|$$

be a closed subset. The property of $Z(u)$ is that $Z(u) = |\Delta|$ if and only if $\chi^u \in \Gamma(X, \mathcal{O}(D))$.

Theorem 0.1. *We have*

$$H^i(X(\Delta), \mathcal{O}(D))_u = H^i_{Z(u)}(|\Delta|; k)$$

where the second term si the singular cohomology of Δ with support in $Z(u)$. For $U = |\Delta| \smallsetminus Z(u)$ we have

$$H^i(X(\Delta), \mathcal{O}(D))_u = \widetilde{H}^{i-1}(U) \quad (i > 0)$$

and that

$$H^0(X(\Delta), \mathcal{O}(D))_u = \begin{cases} k & u \in P_D \\ 0 & \textit{otherwise} \end{cases}.$$

The second part follows from the first part by looking at the long exact sequence of the relative cohomology of the pair $(\Delta, Z(u))$ by noting that $|\Delta|$ is always a contractible subspace of some real space.

COROLLARY 6. *If $|\Delta|$ is convex, and ψ_D is a convex function (then $\mathcal{O}(D)$ is globally generated and) we have*

$$H^i(X(\Delta), \mathcal{O}(D)) = 0 \quad \forall i > 0.$$

PROOF. If ψ_D is convex, then $U = |\Delta| \smallsetminus Z(u)$ is also convex, hence contractible. Therefore $\widetilde{H}^i(U) = 0$ for all $i \geq 0$. □

PROOF OF THE THEOREM. Cover $|\Delta|$ with $\{\sigma_i\}$ for all maximal cones in Δ. We compute the Cech cohomology of $|\Delta|$ with coefficients in k.

$$\begin{array}{lccccccc}
K^\bullet : & \bigoplus_{i_0:\sigma_{i_0}\subset Z(u)} k & \longrightarrow & \bigoplus_{i_0<i_1:\sigma_{i_0}\cap\sigma_{i_1}\subset Z(u)} k & \longrightarrow & \bigoplus k & \longrightarrow & \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \\
C^\bullet(|\Delta|) : & \bigoplus_{\sigma_{i_0}\in\Delta} k & \longrightarrow & \bigoplus_{i_0<i_1:\sigma_{i_0}\cap\sigma_{i_1}\neq\varnothing} k & \longrightarrow & \bigoplus_{i_0<i_1<i_2:\sigma_{i_0}\cap\sigma_{i_1}\cap\sigma_{i_2}\neq\varnothing} k & \longrightarrow & \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \\
C^\bullet(U) : & \bigoplus_{i_0:\sigma_{i_0}\cap U\neq\varnothing} k & \longrightarrow & \bigoplus_{i_0<i_1:\sigma_{i_0}\cap\sigma_{i_1}\cap U\neq\varnothing} k & \longrightarrow & \bigoplus_{i_0<i_1<i_2} k & \longrightarrow & \cdots
\end{array}$$

By definition $K^\bullet$ computes $H^\vee_Z(|\Delta|)$. The claim is that $K^\bullet$ is the degree u part of $C^\bullet(X(\Delta)$. But this follows easily from

$$C^p_u = \prod_{i_0<\cdots<i_p} \mathcal{O}(D)(U_{\sigma_{i_0}} \cap \cdots \cap U_{\sigma_{i_p}}) = \begin{cases} k & u \geq \psi_D \text{ on } \sigma_{i_0} \cap \cdots \cap \sigma_{i_p} \\ 0 & \text{otherwise} \end{cases} .$$

□

1. Rational singularities

Recall that when $f : Y \to X$ is a morphism and $\mathcal{F}$ a sheaf on Y, then $f_*\mathcal{F}$ is a sheaf on X defined by $f_*\mathcal{F}(U) = \mathcal{F}(\widehat{f^{-1}(U)})$. f_* can be right-derived as a functor

$$R^i f_*\mathcal{F}(U) = H^i(f^{-1}\widehat{(U), \mathcal{F}}|_{f^{-1}(U)})$$

DEFINITION 12. Let X be a singular variety and $Y \to X$ a resolution of singularities. Then X has rational singularities if

$$R^0 f_*\mathcal{O}_Y = f_*\mathcal{O}_Y = \mathcal{O}_X, \quad \text{and} R^i f_*(\mathcal{O}_Y) = 0 (\forall i > 0).$$

Example 1.1. If $x \in X$ is an isolated surface singularity and has rational singularities then

$$p(Z) = 1 - \chi(\mathcal{O}_Z) = 0$$

is the arithmetic genus. ⌟

THEOREM 1.1. *All toric varieties have rational singularities.*

PROOF. Given a resolution $f : X(\Delta') \to X(\Delta)$ compute $R^i f_* \mathcal{O}_Y(U_\sigma)$. This coincides with $H^i(X(\sigma'), \mathcal{O}_{X(\sigma')})$. $|\sigma'|$ is convex, and $\mathcal{O}_{X(\sigma')}$ is globally generated therefore $H^i = 0$ for $i > 0$. □

2. Vanishing theorems

If X is toric and complete, and $\mathcal{O}(D)$ is ample (or globally generated) then we have

$$H^i(X, \mathcal{O}(D)) = 0, \quad \forall i > 0.$$

Kodaira vanishing: If X is smooth and projective, $\mathcal{O}(D)$ is ample, then

$$H^i(X, \mathcal{O}(D + K_X)) = 0, \quad \forall i > 0.$$

When X is a smooth variety of $\dim X = n$, and Ω^1_X is the sheaf of regular 1-forms, $\Omega^n_X = \wedge^n \Omega^1_X$ is an invertible sheaf and is by definition $O(K_X)$ where K_X is defined up to linear equivalence. To compute it we can choose a rational section ω of Ω^n_X and set $\mathrm{div}\omega = K_X$.

THEOREM 2.1. *Let $X = X(\Delta)$ is smooth, then $K_X = -\sum_i D_i$ where D_i's range over all irreducible T-invariant divisors.*

PROOF. Basis $v_1, \cdots, v_n$ for N and $v_1^*, \cdots, v_n^*$ for M. Let

$$\omega \in \frac{dx_1}{x_2} \wedge \cdots \wedge \frac{dx_n}{x_n} \in \Omega^n_X(T).$$

ω is a rational section of Ω^n_X and let $\mathrm{div}(\omega) = K_X$. Then we note that if $v_1, \cdots, v_n$ generate U_σ (smooth implies simplicial) we have

$$U_\sigma = \mathcal{S}pec\, k[x_1, \cdots, x_n]$$

and that $\mathrm{div}(\omega) = -D_1 - D_2 \cdots - D_n$ on U_σ and we are done. □

Notice that for $\mathbb{Q}$-divisors (i.e. $D = \sum a_i D_i$ with all $a_i \in \mathbb{Q}$) we get ψ_D which is piecewise linear function not integral. D is ample if ψ_D is strongly convex.

THEOREM 2.2. *Let Δ be a nonsingular complete fan (or $|\Delta|$ is convex). Let D be a $\mathbb{Q}$-divisor on $X(\Delta)$ such that ψ_D is convex. Then*

$$H^i(X, \mathcal{O}(D)) = 0, \forall i > 0.$$

PROOF. Fix $u \in M$ and let $\mathcal{O}(E)$ be a $\mathbb{Z}$-divisor. Let

$$Z(u) = \{u \geq \psi_E\}, \quad U(u) = \{u < \psi_E\}.$$

Then we know that

$$H^i(X(\Delta), \mathcal{O}(E)) = \widetilde{H}^{i-1}(U(u)), i > 0.$$

Since $0 \in Z(u)$, $U(u) \cap S^{n-1}$ is the deformation retract of $U(u)$. Note that $\Delta \cap S^{n-1}$ is a simplicial complex and Δ_u is a subcomplex of $\Delta \cap S^{n-1}$ of simplices in U. Then Δ_u is a deformation retract of U and we have

$$H^i(X(\Delta), \mathcal{O}(E)) = \widetilde{H}^{i-1}(\Delta_u).$$

In the same fashion as in previous vanishing theorems take a convex neighborhood $U_2 \supset \Delta_u$, we get

$$H^i(X, \mathcal{O}(D)) = \widetilde{H}^{i-1}(U_2) = \widetilde{H}^{i-1}(\Delta_u) = \widetilde{H}^{i-1}(U_1) = 0$$

giving us the vanishing. Note that

$$v_i \in U_i \Leftrightarrow u(v_i) < \psi_D(v_i) \in \mathbb{Q} \Leftrightarrow u(v_i) < \lceil \psi_D(v_i) \rceil \Leftrightarrow v_i \in U_2.$$

□

Example 2.1. Let $X = \mathbb{P}^n$. We know that $\operatorname{Pic}(X) = \mathbb{Z} = \{\mathcal{O}(mH) : m \in \mathbb{Z}\}$ fixing a hyperplane H. If $m \geq 0$ we have that $\mathcal{O}(mH)$ is globally generated hence $H^i(\mathbb{P}^n, \mathcal{O}(mH)) = 0$. For $m \in \{-n, -n+1, \cdots, -1\}$ let

$$D = -\varepsilon D_1 - \varepsilon D_2 - \cdots - \varepsilon D_n + (1-\varepsilon) D_{n+1}.$$

Then D is ample for small values of $\varepsilon > 0$ and we have $\lfloor D \rfloor = -D_1 - D_2 - \cdots - D_n \cong -nH$. It follows that $H^i(\mathbb{P}^n, \mathcal{O}(mH)) = 0$ by convexity of ψ_D. ⌟

PROOF OF KODAIRA'S VANISHING THEOREM. If X is complete and D is ample, consider the $\mathbb{Q}$-divisor $E = D + \varepsilon K_X$. Then E is ample if and only if ψ_D is strongly convex. This proves our claim since $\lfloor E \rfloor = D + K_X$. □

THEOREM 2.3 (Kawamata-Viehweg vanishing). *Let D be an ample $\mathbb{Q}$-divisor and X a complete smooth variety. Then*

$$H^i(X, \mathcal{O}(\lceil D \rceil + K_X)) = 0 \quad \forall i > 0.$$

PROOF. $E = D + \varepsilon K$ is ample and $\lfloor E \rfloor = \rfloor D + \varepsilon K \rfloor = \lceil D \rceil + K$. □

3. Serre duality

THEOREM 3.1 (Serre duality). *Let X be n-dimensional and complete and smooth. Let E be the locally free sheaf $\mathcal{O}(D)$. Then we have*

$$H^{n-i}(X, \mathcal{O}(K_X - D)) \cong H^i(X, \mathcal{O}(D))^\vee.$$

The ideal is that this isomorphism is using the fact that this isomorphism is supposed to be preserving the grading by M. From what we know

$$\psi_{K-D} = \psi_K - \psi_D$$

and hence

$$U(u) = \{u < \psi_K - \psi_D\} \supset \Delta_u.$$

Thus $H^{n-i}(X, \mathcal{O}(K_X - D)) = H^{n-i-1}(\Delta_u)$. On the other hand from ψ_D we have $U(-u) = \{-u < \psi_D\} \supset \Delta_{-u}$ and we have

$$H^i(X, \mathcal{O}(D))^\vee_{-u} = \widetilde{H}^{i-1}(\Delta_{-u}).$$

The isomorphism of the above two cohomology groups follows from

$$\widetilde{H}^{i-1}(\Delta_{-u}) = \widetilde{H}_{n-i-1}(\Delta_u) = \widetilde{H}^{n-i-1}(\Delta_u)^\vee.$$

The first isomorphism is Alexander duality and the second one the universal coefficients theorem.

REMARK. In the general case of the Grothendieck-Serre duality for example when X is singular, replace K_X by the dualizing complex $\omega_x^\bullet$. When X is Cohen-Macauly, then $\omega_X^\bullet$ is a sheaf ω_X. This is the case for example for toric varieties as we know. Note that in this case $K_X = \overline{K_{X_s}}$ where X_s is the smooth locus of X.

CHAPTER 3

Special topics

1. Applications of vanishing

Example 1.1. Let P be a lattice polytope. Let $f(m)$ denote the number of points in $mP \cap M$ for all $m \geq 0$. One observes that $f(m)$ is a polynomial in m. Then the claim is that $f(-m)$ is the number of points of $\mathrm{int} mP \cap M$. This was a conjecture of Ehrhardt and then proved by McDonald.

Solution: Assign a fan Δ to P in the usual way. So we have $(X(\Delta), \mathcal{O}(D))$ a polarized toric variety. Then for the ring

$$R = \oplus_{m \geq 0} \Gamma(X(\Delta), \mathcal{O}(mD)) = k[C \cap M\mathbb{Z}]$$

we have that $f(m) = \dim R_m$ is the Hilbert function of R. Which is a polynomial in m for large enough m in general. In fact it is a polynomial in m for all m in the toric case. Because

$$\chi(P(mD)) = \dim H^0(X, \mathcal{O}(mD))$$

happens for all m in the toric case and this is is a polynomial in m because of Riemann-Roch theorem. Therefore $f(-m) = \chi(\mathcal{O}(-mD))$ but by Serre duality we have

$$\begin{aligned}
\chi(\mathcal{O}(-mD)) &= \sum (-1)^i \dim H^i(X, -mD) \\
&= (-1)^{-n} \sum (-1)^{-i+n} \dim H^{n-i}(X, K_X + mD) = (-1)^n \chi(K_X + mD).
\end{aligned}$$

Then $\chi(K_X + mD) = \dim H^0(X, K_X + mD)$ by Kodaira vanishing (otherwise we may resolve Δ first and then pull-back D to $f^*(D)$ and do everything for $X(\Delta')$). Our final observation will be that

$$\dim H^0(K_X + mD) = \#(\text{int } mP \cap M).$$

⌟

2. Gorenstein condition

In general K_X is defined as the dualizing complex of the Serre duality. However when X is normal (and of course singular) we observe that this dualizing element is a sheaf and in fact constructed as the closure of $K_{X^{sm}}$. The next problem is that K_X constructed this way may not be Cartier. But X is called *Gorenstein* if K_X is Cartier.

We have $K_{X^{sm}} = \sum a_i D_i$. This is the case because U_{ρ_i} is completely in the smooth locus of the toric variety therefore $D_i|_{X^{sm}}$ is all of D_i. So we have

$$\overline{K}_{X^{sm}} = \sum a_i \overline{D}_i.$$

So the cases in which this may fail to be Cartier are

(1) There is no plane through v_i,

(2) There is a plane but ψ is not integral and $m.\psi$ is integral $m > 0$.

X is $\mathbb{Q}$-Gorenstein if mK_X is Cartier for some $m > 0$.

Example 2.1. If Δ is simplicial then $X(\Delta)$ is $\mathbb{Q}$-Gorenstein. ⌟

Example 2.2. The fan of the cube centered at the origin is Gorenstein. ⌟

In general we have

PROPOSITION 4. $X(\Delta)$ *is* $\mathbb{Q}$*-Gorenstein if and only if there is a polytope (not necessarily convex) passing through* v_i*'s.*

3. Fano Varieties

Let X be Gorenstein and complete. Then X is Fano if $-K_X$ is ample.

Example 3.1. $\mathbb{P}^n$ is fano since $K_{\mathbb{P}^n} = -(n+1)H$. ⌟

Example 3.2. For a hypersufrace $X \subset \mathbb{P}^n$ of degree d, we have

$$K_X = K_{\mathbb{P}^n} + X|_X = -(n+1)H + dH = (-n-1+d)H.$$

So X is Fano if and only if $-n-1+d < 0$ i.e. $n \geq d$. ⌟

If K_X is $\mathbb{Q}$-Cartier and $-K_X$ is ample then X is called $\mathbb{Q}$-Fano.

3.1. Toric Fano.

If $X(\Delta)$ is Gorenstein, then the condition of being fano is equivalent to $-\psi_{K_X}$ being strictly convex which is equivalent to having Q being strictly convex for any polytope Q constructed from v_i's of each of the maximal cones. Recall that here $Q \subset N$ is the polar dual of P where P is the polytope in $M_{\mathbb{R}}$ associated to $-K_X$. So we have

PROPOSITION 5. *The is a one-to-one correspondence between Fano* $X(\Delta)$ *and pairs of polar dual lattice polytopes* (P, Q)*. The smooth Fano's are in one-to-one correspondence with those pairs* (P, Q) *as above such that the fan over* Q *is nonsingular.*

REMARK. Usually if P is a lattice polytope the the dual $P^{\vee}$ sis not a lattice polytope. P is called *reflexive* if $P^{\vee}$ is again lattice a polytope.

For each dimension d, there is a finite number of nonsiomorphic smooth relfexive polytopes of dimension d. So there is a finite number of isomorphic classes of smooth toric Fano's od fimension d.

If $d = 2$ the Fano variety is also called *del Pezzo surface* which is either $X = \mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{P}^2$ blown-up at ≤ 8 points. Among these there are five toric Fanos

$$fig60$$

If $d = 3$ there are 18 smooth toric fanos and if $d = 4$ there are 123 of them.

4. Singularities

Recall that given $f : Y \to X$ surjective we can pullback the Cartier divisors since if $D \subset \mathrm{CDiv}(X)$ is locally $\mathrm{div}(g)$ then $f^*(D)$ is locally $\mathrm{div}(g \circ f)$. If D is $\mathbb{Q}$-Cartier then $f^*(D)$ is also $\mathbb{Q}$-Cartier. But if D is not Cartier, then $f^*(D)$ is not defined.

Example 4.1. Look at $Bl_0\mathbb{A}^2 \to \mathbb{A}^2$. Then for any D we have

$$f^*D = D' + mE$$

where D' is the strict transform of D and m is the multiplicity of D at zero. ⌟

Let X be a $\mathbb{Q}$-Gorenstein singular variety with K_X being $\mathbb{Q}$-Cartier, and $f : Y \to X$ be a resolution of singularities. The discrepancy of f is

$$K_{Y/X} = K_Y - f^*K_X = \sum a_i E_i.$$

Here E_i's are the exceptional divisors.

Example 4.2. For $f : Bl_0\mathbb{A}^2 \to \mathbb{A}^2$ we have $K_{Y/X} = E$. If X is smooth then $f : Bl_Z X \to X$ has discrepancy

$$K_{Y/X} = (c-1)E$$

where c is the codimension of Z. For this we assume that Z is a nonsingular subvariety. ⌟

We say that X has terminal, canonical, log-terminal and log-canonical singularities if all a_i are respectively positive, non-negative, > -1 and ≥ -1. The key observation is that $\min\{a_i\}$ does not depend on Y if $a_i \geq -1$ in general. If $a_i < -1$ can happen then $\min\{a_i\} = -\infty$.

In the context of the minimal model program one wants to classify the smooth complete (projective) varieties up to birational isomorphism. In dimension 2, this has been carried out. The result is that there is a minimal smooth model after a finite sequence of blow-downs

$$X \to X_1 \to \cdots \to X_{\min}.$$

So $X_{min} \cong Y_{min}$ if X and Y are birational. The only expections are $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$.

Note that terminal singularities is equivalent to being smooth in dimension 2. In dimension greater than two the minimal varieties have terminal singularities. And if X and Y are birational then X_{min} and Y_{min} are related by a sequence of flops. Another way to perform this classification is to use the fact that if X and Y are birational (and of general type) then

$$\Gamma(\mathcal{O}(mK_X)) \cong \Gamma(\mathcal{O}(mK_Y))$$

hence $R = \oplus_{m\geq 0}\Gamma(\mathcal{O}(mK_X))$. Then

$$X_{can} = \mathcal{P}roj\, R = Y_{can}$$

is the canonical model of X.

Example 4.3. In

$$fig61$$

we have $X = U_\sigma$ and $K_X = -D_1 - D_2$. Then $\psi_{K_X}(v_i) = 1$ for $i = 1, 2$ so $\psi_{K_X} = x - \frac{1}{3}y$. We pull this back and get

$$\varphi^*\psi_{K_X}(v_3) = \frac{2}{3}$$

and $f^*(K_X) = -D_1 - D_2 - \frac{2}{3}D_3$. In fact

$$K_{Y/X} = K_Y - f^*K_X = -\frac{1}{3}D_3$$

so X has log-terminal singularity. ⌟

In general if $X = U_\sigma$ and $\mathbb{Q}$-Gorenstein then X is log-terminal. This is easy to se by finding $\psi_{K_Y} - \psi_{K_X}$ one say a v_3 that is between v_1 and v_2 in some certain ordering. So if $K_{Y/X} = \cdots + a_3v_3 + \cdots$ then $a_3 = \ell(v_3) - 1 > -1$ where $\ell(v_3) = \psi_{K_X}(v_3)$. But this is the same as $\psi_{K_Y} - \psi_{K_X}(v_3)$.

If we construct triangles from v_i's and the hyperplane $\ell = 1$. Then X is moreover canonical if there is no lattice points in the triangle (but maybe on the $\ell = 1$ leg), and is terminal if there is no lattice point on the triangle expect at the v_i's and origin.

Remark. In the general (not necessarily toric) case, if X is Gorenstein the $K_{Y/X}$ always has $\mathbb{Z}$-coefficients therefore canonical is equivalent to log-terminal and this is equivalent to having rational singularities.

Definition 13. $Y \to X$ is a crepant resolution if there is no discrepancy: $K_Y = f^*K_X$.

Question: If X is Gorenstein

5. Torus equivariant cohomology

Let Δ be a complete nonsingular fan. Our goal is to compute $H^\bullet(X(\Delta),\mathbb{R})$ from the equivariant cohomology $H_T^\bullet(X(\Delta),\mathbb{R})$. So we will rather look at the analytic topology of X and we restrict ourselves to $k=\mathbb{C}$. All the following constructions are carried out in the analytic category.

In general let T be a group acting freely on X then we define $H_T^\bullet(X) = H^\bullet(X/T)$. Otherwise replace X by $X \times ET$. ET is the universal bundle over BT, the classifying space of T over a point. Note that ET is contractible and T acts freely on ET. At any rate T is acting freely on $X \times ET$ and we define

$$H_T^\bullet(X) := H^\bullet(X \times ET/T).$$

For instance when X is a point then

$$H_T^\bullet(pt) = H^\bullet(ET/T) = H^\bullet(BT).$$

Then induced mapping $H_T^\bullet(pt) \to H_T^\bullet(X)$ from the projection

$$\pi_2 : X \times ET/T \to ET/T$$

provides the structure of a $H_T^\bullet(pt)$-algebra on $H_T^\bullet(X)$.

Example 5.1. Let $T = \mathbb{C}^*$ acting on $\mathbb{C}^n \smallsetminus \{0\}$. We know that $\pi_i(\mathbb{C}^n \smallsetminus \{0\}) = 0$ for all $i < 2n-1$. We have

$$\mathbb{C}^n \smallsetminus \{0\}/T = \mathbb{P}^{n-1}$$

and therefore $H^\bullet(\mathbb{P}^{n-1}) = \mathbb{R}[x]/(x^n)$, with x of degree 2 is the T-equivariant cohomology ring.

For the action of T on $\mathbb{C}^\infty \smallsetminus \{0\}$ we have $\pi_i(\mathbb{C}^\infty \smallsetminus \{0\}) = 0$ for all $i > 0$ so $ET = \mathbb{C}^\infty \smallsetminus \{0\}$. This implies that $H_T^\bullet(pt) = H(\mathbb{P}^\infty) = \mathbb{R}[x]$ and $BT = \mathbb{P}^\infty$.

For $T = (\mathbb{C}^*)^n$ we have $ET = (\mathbb{C}^\infty \smallsetminus \{0\})^n$ and $BT = (\mathbb{P}^\infty)^n$. We get

$$H_T^\bullet(pt) = \mathbb{R}[x_1, \cdots, x_n]$$

where $x_i \in H_T^2(pt)$ are all of degree 2. ⌟

If $H_T^\bullet(X)$ has only even degree cohomology then we recover the cohomology of X from

$$H^\bullet(X) = H_T^\bullet(X)/(x_1,\cdots,x_n).H_T^\bullet(X).$$

In *nice cases*, $H_T^\bullet(X)$ is a free graded $\mathbb{R}[x_1,\cdots,x_n]$-module of the form

$$H_T^\bullet(X) = \oplus e_i k[x_1,\cdots,x_n], \quad e_i \in H_T^{d_i}(X).$$

In this case $\{e_i\}$ forms a basis for $H^\bullet(X)$ as stated.

In the toric case let T be the maximal torus acting on $X(\Delta)$. Let

$$\mathfrak{A}(\Delta) = \{ \text{ continuous piecewise polynomial functions (real-valued) on } \Delta \}.$$

By this we mean that $f : |\Delta| \to \mathbb{R}$ is a continuous function such that $f|_\sigma$ is a polynomial. Then we have

THEOREM 5.1 (Goresky-Kottwitz-MacPherson). *$H^{2i}_T(X(\Delta)) = \mathfrak{A}^i(\Delta)$ when Δ is complete.*

$\mathfrak{A}(\Delta)$ has a grading with $\mathfrak{A}^i(\Delta)$ containing $f \in \mathfrak{A}(\Delta)$ such that $f|_\sigma$ is homogeneous of degree i. Over a point we have A the set of global polynomial functions on $N_\mathbb{R}$:

$$A = \mathbb{R}[x_1, \cdots, x_n].$$

So we have that the T-equivariant cohomology is an A algebra via $A \hookrightarrow \mathfrak{A}(\Delta)$. The real cohomology is given by

$$H^\bullet(\Delta) = \mathfrak{A}(\Delta)/(x_1, \cdots, x_n).\mathfrak{A}(\Delta).$$

Example 5.2. Let $X(\Delta) = \mathbb{P}^1$. We have

$$\mathfrak{A}(\Delta) = \{(f_1, f_2) \in \mathbb{R}[x] : f_1(0) = f_2(0)\} = \mathbb{R}[x] \oplus x\mathbb{R}[x].$$

So we conclude that

$$H^i(X(\Delta)) = \begin{cases} \mathbb{R} & i = 0, 2 \\ 0 & \text{otherwise} \end{cases}.$$

The basis of $(1, 1)$ and $(0, x)$ is the case of H^0 and H^2 respectively. ⌟

To be able to do more interesting computations of the type in the above example we will use Morse theory. Let P be a polytope. The associated fan Δ is simplicial if and only if P is simple (i.e. at every vertex v there are n edges). Let $\ell : P \to \mathbb{R}$ be a continuous function that distinguishes all vertices. Then ℓ will order the vertices $v_1, \cdots, v_m$ and the maximal cones of Δ say as $\sigma_1, \cdots, \sigma_m$. The index of a vertex if

$$\mu(v_i) = \text{ \# edges going down }.$$

Let $\Delta_i = \sigma_1 \cup \cdots \sigma_i$. We compute $\mathfrak{A}(\Delta_i)$ be induction on i. It is a free A-module so we only need to find generators. The claim is that $\mathfrak{A}(\Delta_{i+1}) \xrightarrow{res} \mathfrak{A}(\Delta_i)$ is surjective with a kernel K given by

$$K = \{f \in \mathfrak{A}(\sigma_{i+1}) : f|_{F_j} = 0\}.$$

So K is a free A-module with generators in degree $\mu(v_{i+1})$ and it fits in the trivial extension

$$\mathfrak{A}(\Delta_{i+1}) = \mathfrak{A}(\Delta_i) \oplus K.$$

Our conclusion will be

PROPOSITION 6. *Every vertex v_i contributes a generator of degree $\mu(v_i)$ to $\mathfrak{A}(\Delta)$ and*

$$\dim H^{2k}(X(\Delta)) = \#\{v_i : \mu(v_i) = k\}.$$

COROLLARY 7. *If we replace ℓ by $-\ell$, we get Poicare duality*

$$\dim H^{2k}(X(\Delta)) = \dim H^{2n-2}(X(\Delta)).$$

5.1. Stanley-Reisner ring. Let $\rho_1, \cdots, \rho_d$ be the rays of Δ each generated by vector u_i. Let $\chi_i \in \mathfrak{A}^1(\Delta)$ be piecewise linear:

$$\chi_i(u_j) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

Then the claim is that $\chi_1, \cdots, \chi_d$ generate $\mathfrak{A}(\Delta)$ as an $\mathbb{R}$-algebra. Let $\mathbb{R}^d \to N_{\mathbb{R}}$ be given by $e_i \mapsto u_i$. Then we get a mapping of $\widetilde{\Delta} \to \Delta$ were $\widetilde{\Delta}$ has cones

$$\widetilde{\sigma} = \langle e_{i_1}, \cdots, , e_{im} \rangle$$

for any $\sigma = \langle u_{i_1}, \cdots, u_{i_m} \rangle$. On $\mathbb{R}^d$ the coordinate fucntions $X_1, \cdots, X_d$ will correspond to our $\chi_1, \cdots, \chi_d$. So X_i's generate $\mathfrak{A}(\widetilde{\Delta})$.

$$0 \to I \to \mathbb{R}[\chi_1, \cdots, \chi_d] \to \mathfrak{A}(\Delta) \to 0.$$

PROPOSITION 7. *$I = (\chi_{i_1} \cdots \chi_{i_m})$ were i_j's run over all indices such that ρ_{i_j} do not lie in a cone.*

If $f(\chi_1, \cdots, \chi_d) = 0$ in $\mathfrak{A}(\Delta)$. restrict to $\sigma = \langle \rho_1, \cdots, \rho_n \rangle$ we get $f(\chi_1, \cdots, \chi_n, 0, \cdots, 0)$.

DEFINITION 14. Stanley-Riesner ring is $SR(\Delta) = R/I$.

Example 5.3.

$$H_T^\bullet(\mathbb{P}^2) = \mathbb{R}[\chi_1, \chi_2, \chi_3]/(\chi_1 \cdot \chi_2 \cdot \chi_3).$$

So

$$H^\bullet(\mathbb{P}^2) = \mathbb{R}[\chi_1, \chi_2, \chi_3]/(\chi_1\chi_2\chi_3, \chi_1 - \chi_3, \chi_2 - \chi_3) = \mathbb{R}[\chi_3]/(\chi_3^3).$$

⌟

Example 5.4 (Hard Lefschetz theorem)**.** We can do intersection theory using the above theory. For instance if $[D] = -\psi_D \in \mathfrak{A}(\Delta)$, the Lefschetz operator is

$$L : H^\bullet(X(\Delta)) \to H^\bullet(X(\Delta))$$

is just multiplication by $-\psi_D$. ⌟

Example 5.5 (Face numbers)**.** If Δ is simplicial complete let $h_i = \dim H^{2i}(X)$ be the Betti numbers and let f_k be the number of k-dimensional cones of Δ. Then the two types of numbers are related to each other via

$$P_\Delta(t) = \sum_{i=0}^{n} h_i t^i = \sum_{k=0}^{n} f_k (t-1)^{n-k}.$$

⌟

REMARK. There are conditions that f_i's always satisfy. Stanley [1980] shows that there is a list of relations between f_i's or equivalently h_i's that if guaranteed there is always a polytope P such that $X(\Delta_P)$ yields the given h_i's.

6. Chow groups

Let X be any variety. We have already defined

$$A^1(X) = A_{n-1}(X) = \mathrm{Div}(X)/\sim$$

with $\sim$ being our usual equivalence relation. In general we set

$$A^j(X) = \{\sum a_i Z_i : \ Z_i \subset X \text{ is irreducible of codimension } j\ \}/\sim$$

with the equivalence relation being generated by $Z_1 \sim Z_2$ if there is a flat subvariety $V \subset X \times \mathbb{P}^1$ over $\mathbb{P}^1$ of codimension j such that Z_1 is the geometric fiber over 0 and Z_2 is that of ∞. So except $A^0(X) = \mathbb{Z}[X]$ computing any other Chow group is a hard question.

Our old equivalence relation is a special case of the general one: For any $f \in k(X)$ we have a rational mapping $f : X \to \mathbb{P}^1$. Let V be the graph of f in $X \times \mathbb{P}^1$. Then (show that) V is flat over $\mathbb{P}^1$ and the claim follows.

Chow groups form a ring if X is nonsingular. If we replace the equivalence relation (in the divisors case) with algebraic equivalence (i.e. instead of $\mathbb{P}^1$ using any curve C), we get the connected components of $\mathrm{Pic}(X)$ with is called the Neron-Severi group:

$$NS(X) = \mathrm{Div}(X)/\sim_{alg}.$$

If we use homological equivalence (i.e. $Z_1 \sim Z_2$ if $[Z_1] = [Z_2] \in H^\bullet(X)$) we get a mapping

$$A^\bullet(X) \to H^{2\bullet}(X).$$

This is not injective or necessarily surjective. One case also consider moding out further by numerical equivalence $Z_1 \sim Z_2$ if $Z_1.Y = Z_2.Y$ for all Y.

In the toric case

$$A^\bullet(X) \otimes \mathbb{R} \to H^{2\bullet}(X(\Delta), \mathbb{R})$$

is an isomorphism. For $H^\bullet(X(\Delta))$ is spanned by $[V(\tau)]$ so the above is surjective. Also the relations in cohomology among $\chi_1, \cdots, \chi_d$ are

$$\chi_{i_1} \cdots \chi_{i_m} = 0$$

which correspond to relations in $A^\bullet(X)$: $D_{i_1} \cap \cdots \cap D_{i_m} = \varnothing$ and $\mathrm{div}(\chi^m) = 0$. However the same is true in $A^\bullet(X)$ proving injectivity.

7. Cool facts to recall from intersection theory [Fulton]

• All symmetric polynomial in the Chern roots of E has a definite expression as a polynomial in the Chern classes of E. This includes Todd class and the Chern character. So we start with axiomatic definition of the Chern classes. Find them in terms of the Chern roots of a decomposition and then define the characteristic classes in terms of the Chern roots; e.g.

$$\begin{aligned}\mathrm{td}(E) &= \prod_{i=1}^{s} \frac{\alpha_i}{1-e^{-\alpha_i}} \\ \mathrm{ch}(E) &= \sum e^{\alpha_i}.\end{aligned}$$

• If X is a complex manifold, and $K_0(X)$ is the K-group of vector bundles over X, then the Chern character

$$\mathrm{ch} : K_0(X)_{\mathbb{Q}} \to H^{2\bullet}(X, \mathbb{Q})$$

is a ring homomorphism which is an isomorphism.

• If X is a smooth projective scheme we have a definition for Chern classes and the Chern character of the locally free sheaves on X: The axioms

(1) $L = \mathcal{O}(D)$ then $\mathrm{ch}(L) = e^{[D]} = 1 + [D] + \frac{1}{2}[D][D] + \cdots$;

(2) For exact $0 \to E_1 \to E \to E_2 \to 0$ then $\mathrm{ch}(E) = \mathrm{ch}(E_1) + \mathrm{ch}(E_2)$;

(3) Over flat pullbacks $f^*(\mathrm{ch}(E)) = \mathrm{ch}(f^*(E))$

determine ch(.) uniquely and it coincide with $\sum e^{\alpha_i}$ if $\alpha_i = c_1(L_i)$ where L_i's are the factors of a filteration

$$E^0 \subset E^1 \subset \cdots \subset E^{s-1} \subset E$$

on flat $Y \to X$.

• If $\mathcal{F}$ is a coherent sheaf on X, take a free resolution

$$0 \to E_m \to \cdots \to E_0 \to \mathcal{F} \to 0$$

by locally free sheaves. This exists if X is smooth. Then we define

$$\mathrm{ch}(\mathcal{F}) = \mathrm{ch}(E_0) - \mathrm{ch}(E_1) + \cdots.$$

• If $f : X \to Y$ is proper, have the push-forwards

$$f_*[V] = \begin{cases} \deg(f : V \to f(V)).f(V) & \dim f(V) = \dim V \\ 0 & \text{otherwise} \end{cases}$$

on the Chow groups and on the Grothendieck group of coherent sheaves on X, $K_0(X)_{\mathbb{Q}}$ by

$$f_*(E) = [Rf_*E].$$

• The Grothendieck-Riemann-Roch says that the following diagram is commutative

$$\begin{tikzcd} K_0(X)_{\mathbb{Q}} \arrow[r, "\mathrm{ch}(.).\mathrm{td}_X"] \arrow[d, "f_*"'] & A^\bullet(X)_{\mathbb{Q}} \arrow[d, "f_*"] \\ K_0(Y) \arrow[r, "\mathrm{ch}(.).\mathrm{td}_Y"] & A^\bullet(Y)_{\mathbb{Q}} \end{tikzcd}$$

i.e.

$$f_*(\mathrm{ch}(\mathcal{E}).\mathrm{td}_X) = \mathrm{ch}(Rf_*(\mathcal{E})).\mathrm{td}_Y$$

for any coherent sheaf $\mathcal{E}$ on X.

• If X is singular, $\mathcal{E}$ is locally free we still have a version on the Hirzebruch-Riemann-Roch. Recall that HRR on nonsingular complete variety is the reduction of the above formula to the category of schemes over a point:

$$\chi(\mathcal{E}) = \int_X \mathrm{ch}(\mathcal{E}).\mathrm{td}(X).$$

For singular X let Y be a resolution of singularities. Then we have H-R-R for $\mathcal{E}$ locally free. Since in that case

$$\chi(\mathcal{E}) = \chi(f^*\mathcal{E}) = f^*(\mathrm{ch}(\mathcal{E})) \cap \mathrm{td}_Y = \mathrm{ch}(\mathcal{E}) \cap f_*\mathrm{td}_Y = \mathrm{ch}(\mathcal{E}) \cap \mathrm{td}_X.$$

Remark. If D is a Cartier divisor then $\mathcal{O}(mD)$ is always locally free (really?) so H-R-R generated a proof that $\chi(\mathcal{O}(mD))$ is a polynomial in m:

$$\begin{aligned}\chi(\mathcal{O}(mD)) &= \int \mathrm{ch}(\mathcal{O}(mD)).\mathrm{td}_X = \int e^{m.[D]}(1 + \mathrm{td}_1(X) + \mathrm{td}_2(X) + \cdots) \\ &= \int (1 + m[D] + m^2\frac{[D]^2}{2!} + \cdots + m^d\frac{[D]^n}{n!})(1 + \mathrm{td}_1(X) + \cdots) \\ &= m^n\frac{D^n}{n!} + m^{n-1}\frac{D^{n-1}}{(n-1)!}\mathrm{td}_1 + \cdots\end{aligned}$$

Note that $\mathrm{td}_1 = -K_X$.

8. Toric Riemann-Roch theory

Let P be a lattice polytope and form $(X, \mathcal{O}(D))$. Then $\frac{D^n}{n!}$ is the volume of P. So

$$f(m) = \#(mP \cap M) = \mathrm{Vol}(mP) = m^n\mathrm{Vol}(P).$$

Recall the sheaf of log-pole differentials associated to a divisor $D = \sum D_i$ of normal crossing. Thus locally $D = V(x_1 \cdots x_m)$ in some local coordinates. Then $\Omega^1_X(\log D)$ is locally free with basis $\frac{dx_1}{x_1}, \cdots, \frac{dx_m}{x_m}, dx_{m+1}, \cdots, dx_n$. Recall the short exact sequence

$$0 \to \Omega^1_X \to \Omega^1_X(\log D) \xrightarrow{res} \oplus_i \mathcal{O}_{D_i} \to 0.$$

The projection is $f\frac{dx_i}{x_i} \mapsto f|_{D_i}$.

For $X = X(\Delta)$ and $D = \sum D_i = X \smallsetminus T$ we have

$$\Omega^1_X(\log D) = \mathcal{O}_X \otimes_{\mathbb{Z}} \underbrace{M}_{\mathbb{Z}^n} \cong \mathcal{O}_X^{\oplus n}$$

with a slight abuse of notation for constant sheaves. The isomorphism is given in the reverse direction via

$$f \otimes u \mapsto f\frac{d\chi^u}{\chi^u}.$$

In $K_0(X)$ we therefore have

$$[\Omega_X^1] = [\Omega_X(\log D)] - \sum_{i=1}^{d}[\mathcal{O}_{D_i}].$$

In view of the short exact sequence

$$0 \to \mathcal{I}_{D_i} \to \mathcal{O}_X \to \mathcal{O}_{D_i} \to 0$$

we have

$$[\Omega_X^1] = (n-d)[\mathcal{O}_X] + \sum_{i=1}^{d}[\mathcal{O}(-D_i)].$$

We dualize to

$$[T_X] = (n-d)[\mathcal{O}_X] = \sum[\mathcal{O}(D_i)].$$

We compute the Todd genus and Chern character now. We have $\mathrm{td}(\mathcal{O}(D)) = \frac{D}{1+e^{-D}}$ so

$$\mathrm{td}T_X = \prod_i(1 + \frac{PD[D_i]}{2} + \frac{PD[D_i]^2}{12} + \cdots).$$

For Chern character we have

$$ch_X = \mathrm{ch}(T_X) = \prod_{i=1}^{d}(1 + [PD(D_i)]) = 1 + \sum_i[PD(D_i)] + \sum_{i<j}PD\underbrace{[D_i \cap D_j]}_{V_\tau} + \cdots = \sum_{\tau\in\Delta}[V_\tau].$$

So we also recover $\mathrm{Td}_X = \sum_\tau r_\tau[V_\tau]$ for some $r_\tau \in \mathbb{Q}$. We have

$$\chi(\mathcal{O}(D) = \int_X \frac{D^n}{n!}\mathrm{td}_0 + \frac{D^{n-1}}{(n-1)!}\mathrm{td}_1 + \cdots$$

and we observe that for any $n-i$ dimensional V_τ we have

$$\frac{D^{n-i}}{(n-i)!} \cap [V_\tau] = \frac{(D|_{V_\tau})^{n-i}}{(n-i)!} = \mathrm{Vol}P_\tau.$$

So our combinatorial Riemann-Roch theorem is

$$\#(mP \cap M) = \chi(\mathcal{O}(mD)) = \sum_\tau r_\tau.\mathrm{Vol}(P_\tau) = m^n\mathrm{Vol}(P) + \frac{m^{n-1}}{2}\sum_{\text{facets}}\mathrm{Vol}(P_\tau) + \cdots.$$

Example 8.1 (Pick's theorem)**.** For surfaces we have

$$\#(mP \cap M) = m^2\mathrm{area}(P) + m/2\sum_{\text{edges}}\mathrm{length}(E) + \int \mathrm{td}_2(X).$$

We do not need to compute the last term since $1 = \chi(\mathcal{O}_X)$ in the case $m = 0$ and thus $\mathrm{td}_2(X) = 1$. We conclude that

$$\#(P \cap M) = \mathrm{area}(P) + \frac{1}{2}\mathrm{perimeter}(P) + 1.$$

⌟

Example 8.2. For $\mathcal{O}(D) = \mathcal{O}(d+1)$ in $\mathbb{P}^3$ we get a regular polytope of edge length d and by the above we have

$$\#P \cap M = \frac{d^3}{6} + d^2 + \frac{11}{6}d + 1.$$

⌟